On some Generalized Nörlund Ideal Convergent Sequence Spaces

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**ABSTRACT**

In this paper, some new Ideal convergent sequence spaces \( c_{(N,p,q)}^i \), \( (c_0)_{(N,p,q)}^i \) and \( (\ell_\infty)_{(N,p,q)}^i \) that are related to the \((N,p,q)\) - summability method, are introduced and some topological properties of these spaces and some inclusion relations and results are determined.

**1. INTRODUCTION**

We denote the space of all real valued sequences by \( \omega \). Each vector subspace of \( \omega \) is called as a sequence space as well. The spaces of all bounded, convergent and null sequences are denoted by \( \ell_\infty \), \( c \) and \( c_0 \), respectively. By \( \ell_1 \), \( \ell_p \), \( cs \), \( cs_0 \) and \( bs \), we denote the spaces of all absolutely convergent, \( p \)-absolutely convergent, convergent, convergent to zero and bounded series, respectively; where \( 1 < p < \infty \).

A linear topological space \( \lambda \) is called a K-space if each of the map \( p_i : \lambda \rightarrow \mathbb{C} \) defined by \( p_i(x) = x_i \) is continuous for all \( i \in \mathbb{N} \), where \( \mathbb{C} \) denotes the complex field and \( \mathbb{N} = \{0,1,2,3,\ldots\} \). A K-space \( \lambda \) is called an FK-space if \( \lambda \) is a complete linear metric space. If an FK-space has a normable topology then it is called a BK-space, (ABFB 2005). If \( \lambda \) is an FK-space, \( \Phi \subset \lambda \) and \( \{e^k\} \) is a basis for \( \lambda \) then \( \lambda \) is said to have AK property, where \( \{e^k\} \) is a sequence whose only term in \( k^{th} \) place is 1 the others are zero for each \( k \in \mathbb{N} \) and \( \Phi = \text{span}\{e^k\} \). If \( \Phi \) is dense in \( \lambda \), then \( \lambda \) is called AD-space, thus AK implies AD.

Let \( \lambda \) and \( \mu \) be two sequence spaces, and \( A = (a_{nk}) \) be an infinite matrix of real or complex numbers, where \( n,k \in \mathbb{N} \). For every sequence \( X = (X_k^n) \in \lambda \) the sequence \( Ax = Ax = ((Ax)_n^k) \in \mu \) is called A-transform of \( x \), where

\[
(Ax)_n^k = \sum_{k=0}^{\infty} a_{nk} x_k. \quad (1)
\]

Then, a defines a matrix mapping from \( \lambda \) to \( \mu \) and we show it by writing \( A : \lambda \rightarrow \mu \).

By \( A \in (\lambda : \mu) \), we denote the class of all matrices \( A \) such that \( A : \lambda \rightarrow \mu \) if and only if the series on the right side of (1) converges for each \( n \in \mathbb{N} \) and every \( x \in \lambda \), and we have \( Ax = ((Ax)_n^k) \) belongs to \( \mu \) for all \( x \in \lambda \). A
sequence $x$ is said to be A-summable to $l$ and is called as the A-limit of $x$.

Let $\lambda$ be a sequence space and $A$ be an infinite matrix. The matrix domain $\lambda_A$ of $A$ in $\lambda$ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}$$

Which is a sequence space.

Let $(t_k)$ be a nonnegative real sequence with $t_0 > 0$ and $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$.

Then, the Nörlund mean with respect to the sequence $t = (t_k)$ is defined by the matrix $N^t = (a^t_{nk})$ as follows

$$a^t_{nk} = \begin{cases} \frac{t_n - t_k}{t_n}, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

for every $k, n \in \mathbb{N}$. It is know that the Nörlund matrix $N^t$ is a Toeplitz matrix if and only if $\frac{t_n}{T_n} \to 0$ as $n \to \infty$. Furthermore, if we take $t = \varepsilon = (1, 1, 1, \ldots)$, then the Nörlund matrix $N^\varepsilon$ is reduced to Cesàro mean $C$ of order one and if we choose $t_n = A^{n-1}_{nk}$ for every $n \in \mathbb{N}$, then the $N^t$ Nörlund mean becomes Cesàro mean $C$, for every $r > 1$ and

$$A^\varepsilon_n = \frac{(r+1)(r+2)\ldots(r+n)}{n!}, \quad n = 1, 2, 3, \ldots$$

Let $t_0 = D_0 = 1$ and define $D_n$ for $n \in \{1, 2, 3, \ldots\}$ by

$$D_n = \begin{bmatrix} t_1 & 1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & 1 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & 1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 \end{bmatrix}$$

With $D_1 = t_1$, $D_2 = (t_1)^2 - t_2$, $D_3 = (t_2)^3 - 2t_1t_2 + t_3$, \ldots

then the inverse matrix $U^t = (u^t_{nk})$ of Nörlund matrix $N^t$ was defined by Mears in (MFM 1943) for all $n \in \mathbb{N}$ as follows

$$u^t_{nk} = \begin{cases} (-1)^{n-k}D_{n-k}T_k, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

**Definition 1.1.** A family $I \subset 2^X$ of subset of a nonempty set $X$ is said to be an ideal in $X$ if

i) $\emptyset \in I$,

ii) For $A, B \in I$ imply $A \cup B \in I$,

iii) $A \in I, B \subset A$ imply $B \in I$.

The ideal $I$ of $X$ is said to be non-trivial if and only if $I \neq 2^X$. The non-trivial ideal $I \subset 2^X$ is called an admissible ideal in $X$ if and only if it contains $\{y \in X : \exists y \in X\}$.

A non-trivial ideal $I$ is called maximal if there cannot exist any non-trivial ideal $J \neq I$ containing $I$ as a subset.

**Definition 1.2.** Let $I \subset 2^X$ be an ideal on $X$. The non-empty family of sets $F(I) \subset 2^X$ is called Filter on $X$ corresponding to $I$ if and only if

i) $\emptyset \notin F(I)$,

ii) For $A, B \in F(I)$ imply $A \cup B \in F(I)$,

iii) For each $A \in F(I)$ and $A \subset B$ implies $B \in F(I)$.

For each ideal $I$, there is a Filter $F(I)$ corresponding to $I$, that is, the following set $F(I)$ is called filter according to the ideal $I$:

$F(I) = \{K \subset 2^X : K^2 \in I\}$

where $K^2 = X \setminus K = X - K$.

**Definition 1.3.** The sequence $x = (x_n)_{n \in \mathbb{N}} \in \omega$ is called ideal convergent or $I$-convergent to a number $L$ if for every $\varepsilon > 0$
A(ε) = \{n \in \mathbb{N} : |x_n - L| \geq ε \} \in I

And if is denoted by

I - \lim x_n = L.

The space of all I-convergent sequences to L is denoted by \(c^1\) as follow;

\[ c^1 = \{x = (x_k) \in w : \{k \in \mathbb{N} : |x_k - L| \geq ε \} \in I\} \]


**Definition 1.4.** The sequence \(x = (x_n)_{n \in \mathbb{N}} \in w\) is said to be I-null if \(L = 0\). In this case it is denoted by

I - \lim x_n = 0

The space of all I-null sequences is defined by \(c^1_0\) as

\[ c^1_0 = \{x = (x_k) \in w : \{k \in \mathbb{N} : |x_k| \geq ε \} \in I\} \]


**Definition 1.5.** A sequence \(x = (x_n)_{n \in \mathbb{N}} \in w\) is said to be I-bounded if there exist a real constant \(M \geq 0\) such that

\[ \{k \in \mathbb{N} : |x_k| \geq M \} \in I \]

(TBC 2005)

**Definition 1.6.** Let X be a linear space. A function \(g: X \to \mathbb{R}\) is called a paranorm if for all \(x,y,z \in X\);

1) \(g(x) = 0\) if \(x = \theta\),
2) \(g(-x) = g(x)\),
3) \(g(x + y) \leq g(x) + g(y)\),
4) If \((\lambda_n)\) is a sequence of scalars with \(\lambda_n \to \lambda(n \to \infty)\) and \(x_n,L \in X\) with \(x_n \to L(n \to \infty)\) in the sense that \(g(x_n - L) \to 0(n \to \infty)\), in the sense that \(g(\lambda_n x_n - \lambda L) \to 0(n \to \infty)\).

**Definition 1.7.** A sequence space \(X\) is called solid or normal if \(x = (x_k) \in X\) implies \(\alpha x = (\alpha_k x_k) \in X\) for all sequence of scalars \(\alpha = (\alpha_k)\) with \(|\alpha_k| < 1\) for all \(k \in \mathbb{N}\). (TBC 2005)

**Definition 1.8.** A sequence space \(X\) is called monotone if it contains the canonical pre-images of all its step-spaces, (TBC 2005)

Let \(K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}\) and \(E\) be a sequence space. A K-step space of \(E\) is a sequence space \(\lambda^K_E = \{(x_k) \in w : (x_n) \in E\}\). A canonical preimage of a sequence \(x_{k_n} \in \lambda^K_E\) is a sequence \(y = (y_n) \in w\) defined as

\[ y_n = \begin{cases} x_n & \text{if } n \in K \\ 0 & \text{otherwise} \end{cases} \]

A canonical preimage of step space \(\lambda^K_E\) is a set of canonical preimage of all the elements in \(\lambda^K_E\) if and only if it is a canonical preimage of some \(x \in \lambda^K_E\) see (HBT 2014).

**Lemma 1.9.** The sequence space \(X\) is solid implies that \(X\) is monotone, (see KPK 2009 p.53).

**2. GENERALIZED WEIGHTED NORLUND IDEAL CONVERGENCE**

Let \(p = (p_k)\) and \(q = (q_k)\) be two increasing sequences of non-zero real constant which satisfy

\[ P_n = p_1 + p_2 + \cdots + p_n, P_{-1} = p_{-1} = 0, \]
\[ Q_n = q_1 + q_2 + \cdots + q_n, Q_{-1} = q_{-1} = 0 \]

Now, we define the Cauchy product of the sequences \(P_n\) and \(Q_n\), as follow
Then, the series $\sum x_k$ or any sequence $x = (x_k)$ is summable to any point $L$ by generalized Nörlund method which is denoted by $x_k \to L(N,p,q)$ if

$$\lim_{n \to \infty} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} x_k = L.$$ 

This is obvious that when we take $p_n = 1$ for each $n \in \mathbb{N}$, then we Nörlund method. (See OTFB 2016). Since we take $p_n = q_n = 1$ for each $n \in \mathbb{N}$, then we approach Cesaro method.

The matrix $A = (\alpha_{nk})$ in $(N,p,q)$-summability is defined by

$$\alpha_{nk} = \begin{cases} p_k q_{n-k} / R_n, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}$$

In this paper, we construct the new I-convergent sequence spaces related to the $(N,p,q)$-summability method. Now, by $c_{(N,p,q)}^I$, $(c_0)_{(N,p,q)}^I$ and $(\ell_{\infty})_{(N,p,q)}^I$, we define generalized weighted Nörlund I-convergent, generalized weighted Nörlund I-null and generalized weighted Nörlund I-bounded sequence spaces, respectively. First we give some topological properties of these spaces. Then, we derive some inclusion relations and results.

A sequence $x = (x_k)$ is said to be generalized weighted Nörlund ideal convergent if for every $\varepsilon > 0$

$$M(\varepsilon) = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L| \geq \varepsilon \right\} \in I$$

And the set of all generalized weighted Nörlund $I$—convergent, generalized weighted Nörlund $I$—null and generalized weighted Nörlund $I$—bounded sequence spaces are defined as follows ;

$$c_{(N,p,q)}^I = \left\{ x = (x_k) \in \omega : \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L| \geq \varepsilon \right\} \in I \right\}$$

$$c_{(N,p,q)}^{I,0} = \left\{ x = (x_k) \in \omega : \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| \geq \varepsilon \right\} \in I \right\}$$

$$c_{(N,p,q)}^{I,\infty} = \left\{ x = (x_k) \in \omega : \left\{ n \in \mathbb{N} : \exists M > 0 \exists \varepsilon > 0 \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| > M \right\} \in I \right\}$$

**Theorem 2.1.** The spaces $c_{(N,p,q)}^I$, $(c_0)_{(N,p,q)}^I$, $(\ell_{\infty})_{(N,p,q)}^I$ are linear spaces

**Proof.** We shall prove the result for the space $c_{(N,p,q)}^I$. Let $x = (x_k), y = (y_k) \in c_{(N,p,q)}^I$ and $\alpha, \beta \in \mathbb{C}$ are given. Then we have the following for given every $\varepsilon > 0$

We denote

$$A(\varepsilon) = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L_1| \geq \varepsilon \right\} \in I$$

$$B(\varepsilon) = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |y_k - L_2| \geq \varepsilon \right\} \in I$$

for some $L_1, L_2 \in \mathbb{C}$.

Now, we write the following inequality

$$\frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} (\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)$$

$$\leq \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} (|\alpha| |x_k - L_1| + |\beta| |y_k - L_2|)$$

$$\leq |\alpha| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L_1| + |\beta| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |y_k - L_2|$$

Then, by using the above inequality we derive
\[ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} (\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2) \geq \varepsilon \]
\[
\leq n \in \mathbb{N} : |\alpha| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L_1| \geq \frac{\varepsilon}{2}
\]
\[
\cup \left\{ n \in \mathbb{N} : |\beta| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |y_k - L_2| \geq \frac{\varepsilon}{2} \right\}
\]
\[
\subseteq A(\varepsilon) \cup B(\varepsilon) \subseteq \mathbb{N}
\]

Then this completes the proof. The proof for the spaces \( c^l_{(N,p,q)} \) and \((l_\infty)^l_{(N,p,q)}\) follow similarly.

**Theorem 2.2.** The spaces \( c^l_{(N,p,q)} \), \((c_0)^l_{(N,p,q)}\), \((l_\infty)^l_{(N,p,q)}\) are para-normed spaces with the para-norm
\[
g(x) = \sup_{n \in \mathbb{N}} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k|
\]

**Proof.** Since we have similar proof for \( c^l_{(N,p,q)} \), \((c_0)^l_{(N,p,q)}\), \((l_\infty)^l_{(N,p,q)}\), we give only the proof for \( c^l_{(N,p,q)} \). It is trivial that if \( x = (x_k) = 0 \) then \( g(x) = 0 \). For \( x = (x_k) \neq 0 \) then \( g(x) \neq 0 \), we have that

i) For all \( x \in c^l_{(N,p,q)} \)
\[
g(x) = \sup_{n \in \mathbb{N}} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| \geq 0
\]

ii) For all \( x \in c^l_{(N,p,q)} \)
\[
g(-x) = \sup_{n \in \mathbb{N}} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |-x_k| = \sup_{n \in \mathbb{N}} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| = g(x)
\]

iii) For every \( x, y \in c^l_{(N,p,q)} \)
\[
g(x + y) = \sup_{n \in \mathbb{N}} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - y_k|
\]
\[
\leq \sup_{n \in \mathbb{N}} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| + \sup_{n \in \mathbb{N}} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |y_k|
\]
\[
= g(x) + g(y).
\]

iv) Let \((\lambda_n)\) is a sequence of scalars with \( \lambda_n \to \lambda (n \to \infty) \) and \( x_n \in c^l_{(N,p,q)} \)

\[
\frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| \to L(n \to \infty),
\]

in the sense that
\[
g\left( \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| \to L \right) \to 0(n \to \infty).
\]

Therefore,
\[
g\left( \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| - \lambda L \right) \leq g\left( \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| (\lambda_n - \lambda) \right)
\]
\[
+ g\left( \lambda \left( \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| - L \right) \right)
\]

Then it is obvious that
\[
\lambda_n \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| \to \lambda L(n \to \infty).
\]

This completes the proof.

**Theorem 2.3.** The space \( c^l_{(N,p,q)} \) is solid and monotone.

**Proof.** Suppose that \( x = (x_k) \in c^l_{(N,p,q)} \) and \((a_k)\) be a sequence of scalars with \(|a_k| \leq 1\) for all \( k \in \mathbb{N} \). Then notice that
Furthermore,

\[
\left(12\right) \quad \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |\alpha_k x_k| \geq \varepsilon \right\} 
\subseteq \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| \geq \varepsilon \right\}
\]

Then by using (12) we derive \((\alpha_k x_k) \in c^I_{(N,p,q)}\). This completes the proof.

**Theorem 2.4.** \(c^I_{(N,p,q)}\) is a closed subset of \((l_\infty)_{(N,p,q)}\).

**Proof.** Let’s take a Cauchy sequence \(x^{(n)}_k\) in \(c^I_{(N,p,q)}\) such that \(x^{(n)}_k \to x\) as \(n \to \infty\). We need to show that \(x \in c^I_{(N,p,q)}\). Since \(x^{(n)}_k \in c^I_{(N,p,q)}\) then there exist a sequence of complex number \(\alpha_n\) such that

\[
A = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x^{(n)}_k - \alpha_n| \geq \varepsilon \right\} \in I
\]

(13)

Now, to give the proof, we need to mention that \(\alpha_n \to x\) as \(n \to \infty\) \((A')^c \in I\) whenever

\[
A' = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - a| \geq \varepsilon \right\}
\]

Since \(x^{(n)}_k\) is a Cauchy sequence in \(c^I_{(N,p,q)}\), we can write for a given \(\varepsilon > 0\), there exist \(k_0 \in \mathbb{N}\) such that

\[
\frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x^{(m)}_k - x^{(m)}_k| \leq \frac{\varepsilon}{3} \quad \text{for all } m,n \geq k_0
\]

Let us define the followings sets for \(\varepsilon > 0\) as:

\[
A_1 = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x^{(n)}_k - x^{(m)}_k| < \frac{\varepsilon}{3} \right\}
\]

\[
A_2 = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x^{(n)}_k - \alpha_n| < \frac{\varepsilon}{3} \right\}
\]

\[
A_3 = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x^{(n)}_k - \alpha_m| < \frac{\varepsilon}{3} \right\}
\]

For all \(m,n \geq k_0\) whenever \(A_1^c, A_2^c, A_3^c \in I\). Then we have

\[
\left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |\alpha_n - \alpha_m| < \varepsilon \right\} \supseteq \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x^{(n)}_k - x^{(m)}_k| < \frac{\varepsilon}{3} \right\}
\]

\[
\cap \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x^{(n)}_k - \alpha_n| < \frac{\varepsilon}{3} \right\}
\]

\[
\cap \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x^{(n)}_k - \alpha_m| < \frac{\varepsilon}{3} \right\}
\]

We can see that \((\alpha_n)\) is a Cauchy sequence in \(C\) and convergent to the scalar \(\alpha\) as \(n \to \infty\).

Now, for the last needed let’s take \(0 < \delta < 1\). Then we need to show that if

\[
A' = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - a| < \varepsilon \right\}
\]

Then \((A')^c \in I\). Since

\[
\frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x^{(n)}_k - x_k| \to 0 \quad \text{as } n \to \infty,
\]

then there exists \(n_0 \in \mathbb{N}\) such that

\[
E_1 = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x^{(n)}_k - x_k| < \frac{\delta}{3} \right\}
\]
Which implies that $(E_1)^c \in I$ for all $n \geq n_0$. And we already have from the first part that
\[
E_2 = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - a| < \frac{\delta}{3} \right\}
\]
Which gives us $(E_2)^c \in I$ for all $n \geq n_0$. Since the set $A \in I$ defined as in (13) $\delta$ instead of $\varepsilon$, then we have a subset $E_3 \subset \mathbb{N}$ such that $(E_3)^c \in I$ whenever,
\[
E_3 = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k^{(n)} - a_n| < \frac{\delta}{3} \right\}
\]
Then we may easily say that $(A^n)^c \supseteq E_1 \cap E_2 \cup E_3$. Then by the definition of filter on the ideal that we can say $C_{(N,p,q)}^I \subseteq (l_{\infty})^I$. This completes the proof.

**Theorem 2.5.** The inclusions $(c_0)^I \subseteq c_{(N,p,q)}^I \subseteq (l_{\infty})^I$ are proper.

**Proof.** Let’s take a sequence $x = (x_k) \in (c_0)^I$. Then we have
\[
\{ n \in \mathbb{N} : |x_n| \geq \varepsilon \} \in I
\]
Since $c_0 \subseteq c_{(N,p,q)} \subseteq l_{\infty}$ which give us that $x = (x_k) \in c_{(N,p,q)}^I$ implies
\[
\left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L| \geq \varepsilon \right\} \in I
\]
Now, let us define the following sets
\[
A_1 = \{ n \in \mathbb{N} : |x_n - L| < \varepsilon \}
\]
\[
A_2 = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L| < \varepsilon \right\}
\]
Such that $A_1^c, A_2^c \in I$. Since
\[
\ell_{\infty} = \{ x = (x_n) \in \omega : \sup_{n} |x_n| < \infty \}
\]
When we take supremum over $n$ then we get $A_1^c \subseteq A_2^c$. Then we conclude as $(c_0)^I \subseteq c_{(N,p,q)}^I \subseteq (l_{\infty})^I$.

**REFERENCES**


