

# RESEARCH PAPER

## Category of Source Proper Group Space

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### ABSTRACT:

Our main interest in this study is to look for group space and the topological groupoid. In this paper, we give some results of group space and the topological groupoid, which are properties source proper group space ( $Sp\Gamma$  – space). Finally, remarks, proportions, theorems and example are our own unless otherwise referred.

KEY WORDS: Category, Groupoid, Topological groupoid, Topological group, Group space, source proper groupoid, source proper group space, Submersive group space.  
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### INTRODUCTION :

In this paper, we study the basic construction of group spaces and we divide into three sections. In section one, we give the definitions and theorems about categories, fiber product and proper mappings. And we introduce definitions and theorems about groupoid and morphism of groupoids [Hameed, 2007].

In section two, we introduce a topological group and a topological space in order to define a group space, new types of group spaces are introduced which are source proper group space ( $Sp\Gamma$  – space) and give some of their properties while in section three, we construct submersive group spaces and give some of their properties.

### 2- Category, fiber product and proper map, Groupoid, topological groupoid.

#### 2.1 Definition [Mucuk,1997]:

A category  $C$  consist of:

- (a) A class of objects.
- (b) For every ordered pair of objects  $X$  and  $Y$ , a set  $\text{hom}(X, Y)$  of morphism with domain  $X$  and range  $Y$ , if  $f \in \text{hom}(X, Y)$  we write  $f: X \rightarrow Y$ .
- (c) For every ordered triple of objects  $X, Y$ , and  $Z$  a function  $a$  associating to a pair of morphism  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  their composite  $gf: X \rightarrow Z$ . these satisfy the following two axioms:
  - (i) Associativity: if  $f: X \rightarrow Y$ ,  $g: Y \rightarrow Z$ , and  $h: Z \rightarrow W$  then  $h(gf) = (hg)f$ .
  - (ii) Identity: for every object  $y$  there is a morphism  $I_Y: Y \rightarrow Y$  such that if  $f: X \rightarrow Y$ , then  $I_Y \circ f = f$ , and if  $h: Y \rightarrow Z$ , then  $h \circ I_Y = h$ .

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**2.2 Example of categories [Mucuk,1997], [AL-Taai,1998]:**

The category of topological spaces and continuous maps which we denote by T.

**2.3 Definition [Bredon 1972],[ Dindonne ,1972]:**

Let  $X, Y$  and  $Z$  be topological spaces and let  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$  be continuous maps. Then the fiber product of  $f$  and  $g$  is  $X \times_Y = \{(x, y): f(x) = g(y)\}$  which is a subspace of  $X \times Y$ .

**2.4 Definition [Bourbaki,1989]:**

Let  $f$  be a mapping of a topological space  $X$  into topological spaces  $Y$ . Then  $f$  is said to be proper if  $f$  is continuous and the mapping  $f \times I_Z: X \times Z \rightarrow Y \times Z$  is closed for every topological spaces  $Z$ .

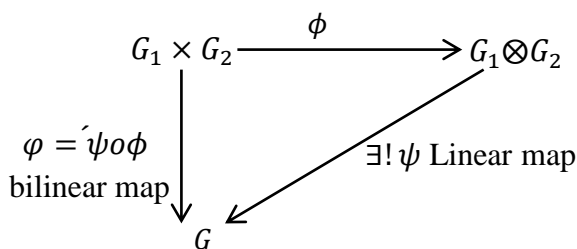
**2.5 Definition [Husmoller ,1966]:**

Let  $G_1$  and  $G_2$  be groupoids spaces the Tensor product of  $G_1 \otimes G_2$  consist of Linear combinations of elements of the form  $g_1 \otimes g_2$  where  $g_1 \in G_1$  and  $g_2 \in G_2$  with the following relations:

- (i)  $C(g_1 \otimes g_2) = (cg_1) \otimes g_2 = g_1 \otimes (cg_2)$  for any scalars  $c$ .
- (ii)  $g_1 \otimes g_2 + g'_1 \otimes g_2 = (g_1 + g'_1) \otimes g_2$ .
- (iii)  $(g_1 \otimes g_2) + (g_1 \otimes g'_2) = g_1 \otimes (g_2 + g'_2)$  for ever  $g_1, g'_1 \in G_1$  and  $g_2, g'_2 \in G_2$ .

**2.6 Definition [Majeed\*,2018]:**

Let  $G_1$  and  $G_2$  be any groupoid spaces then the tensor product  $G_1 \otimes G_2$  exists. For there more if  $\varphi$  is any bilinear map of  $G_1 \times G_2$  into  $G$  then there exists a unique linear map, such that the following diagram commutes, this property of  $G_1 \otimes G_2$  is called universal property of tensor product , where  $G$  is a groupoid space.



**2.7 Proposition [Bourbaki,1989, corollary, page 100]:**

Let  $f: X \rightarrow Y$  be a proper map, then the restriction of  $f$  to a closed subset  $A$  of  $X$  is a proper mapping of  $A$  into  $Y$ .

**2.8 Proposition [Bourbaki,1989, proposition 5, page 99]:**

Let  $f: x \rightarrow y$  and  $g: y \rightarrow z$  be continuous maps then:

- (i) If  $f$  and  $g$  are proper then  $gof$  is proper.
- (ii) If  $gof$  is proper and  $f$  is surjective then  $g$  is proper.
- (iii) If  $gof$  is proper and  $g$  is injective then  $f$  is proper.
- (iv) If  $gof$  is proper and  $y$  is a Hausdorff space then  $f$  is proper.

**2.9 Proposition [Bourbaki,1989, proposition 4, page 98]:**

Let  $f_1: x_1 \rightarrow y_1$  and  $f_2: x_2 \rightarrow y_2$  be continuous maps then:

- (i) If  $f_1$  and  $f_2$  are proper then  $f_1 \times f_2$  is proper.
- (ii) If  $f_1 \times f_2$  is proper and  $x_i \neq \emptyset, i = 1, 2$  then  $f_1$  and  $f_2$  are proper.

Recall that a topological spaces  $x$  is called a compact space if every open cover of  $x$  has a finite sub cover [30],[25]. Group of Bourbaki in[3] defines a compact space with additional property that  $x$  is a Hausdorff space.

**2.10 Definition [Mackenzie,1987],[Moerdijk and Crainic ,2000]:**

A groupoid is a pair of sets  $(G, B)$  on which are given :

- (i) Two surjections  $\alpha, \beta: G \rightarrow B$  called the source and the target map respectively.
- (ii) An injection  $W: B \rightarrow G$  called the unit with  $\alpha o W = I_B, \beta o W = I_B$  where  $I_B: B \rightarrow B$  is the identity map.
- (iii) A law of partial composition  $\gamma$  in  $G$  defined as a law of composition on the set  $G * G = \{(g, g') \in G \times G \mid \alpha(g) = \beta(g')\}$  "fiber product of  $\alpha$  and  $\beta$  over  $B$ ".

Such that:

- (a)  $y(g, y(g_1, g_2)) = y(y(g, g_1), g_2), \forall (g, g_1), (g_1, g_2) \in G * G$ .

- (b)  $\alpha(y(g_1, g_2)) = \alpha(g_2), \beta y(g_1, g_2) = \beta(g_1)$  for each  $(g, g_1) \in G * G$ .
- (c)  $y(g, w(\alpha(g))) = g$  and  $y(w(\beta(g)), g) = g, \forall g \in G$ .
- (iv) A bijection  $\sigma: G \rightarrow G$  called the inversion of  $G$  satisfying:
  - (a)  $\alpha(\sigma(g)) = \beta(g), \beta(\sigma(g)) = \alpha(g), \forall g \in G$ .
  - (b)  $y(\sigma(g), g) = w(\alpha(g)), y(g, \sigma(g)) = w(\beta, g), \forall g \in G$ .

- We write  $(g) = g^{-1}$ , called the inverse element of  $g \in G$  and  $w(x) = \bar{x}$  called the unit element in  $G$  associated to the element  $x \in B$ . Also, we write  $y(g, \dot{g}) = g\dot{g}$ .  $G$  is called the groupoid and  $B$  is called the base. Also, we say that  $G$  is a groupoid on  $B$ .

**2.11 Definition [Mackenzie,1987],[Spaier,1966]:**

A topological groupoid is a groupoid  $(G,B)$  together with topologies on  $G$  and  $B$  such that the maps  $\alpha, \beta, w, y$  and  $\sigma$  are continuous where  $G * G$ . A topological sub groupoid is subgroupoid  $(H,A)$  with the subspace topology from  $(G,B)$ .

A morphism of topological groupoid is a morphism of groupoids  $(f, f_0): (G, B) \rightarrow (\hat{G}, \hat{B})$  such that  $f$  and  $f_0$  are continuous. An isomorphism of a topological groupoid is a morphism of topological groupoids such that  $f: G \rightarrow \hat{G}$  is a homeomorphism. We denote by TG the category whose objects are topological groupoids and whose morphisms are continuous groupoid morphisms.

**3- Topological group , Group Spaces, source prop groupoid, source proper space.**

**3.1 Definition [Higgins,1979],[ IIIman,2002]:**

A topological group is a set  $\Gamma$  with two structures:

- (i)  $\Gamma$  is a group.
- (ii)  $\Gamma$  is a topological space.

Such that the two structures are compatible that is the multiplication law  $M: \Gamma \times \Gamma \rightarrow \Gamma$  and the inversion law  $V: \Gamma \rightarrow \Gamma$  are both continuous where  $\Gamma \times \Gamma$  carries the product topology. A sub set  $H$  of  $\Gamma$  is called topological subgroup if  $H$  is a subgroup and subspace of  $\Gamma$ .

**3.2 Definition [IIIman,2002]:**

Let  $\Gamma$  be a topological group and  $E$  be a topological space. A left action of  $\Gamma$  on  $E$  is a continuous map  $\varphi: \Gamma \times E \rightarrow E$  such that:

- (i)  $\varphi(e, z) = z$ , for all  $z \in E$ , where  $e$  is the identity element in  $\Gamma$ .
- (ii)  $\varphi(r, \varphi(h, z)) = \varphi(M(r, h), z), \forall z \in E$ , where  $M$  is a law of multiplication of  $\Gamma$ .

The space  $E$  together with action  $\varphi$  is called a group space and denoted by  $\Gamma - \text{space}$  more precisely(left  $\Gamma - \text{space}$ ). In a similar way one can define a right  $\Gamma - \text{space}$ .

**3.3 Definition [IIIman,2002],[ Bredon,1972]:**

Let  $E$  be a  $\Gamma - \text{space}$  then:

- (i) The orbit of  $z \in E$  is defined to be the set  $\text{Orb}(z) = \varphi(z, \Gamma) = \{\varphi(z, r) : r \in \Gamma\}$  and the set of orbits of  $E$  is called an orbit space and denoted by  $E/\Gamma$ .

**3.4 Theorem [Bredon,1972 , theorem 3.1 , page 38]:**

Let  $E$  be a  $\Gamma - \text{space}$  with  $\Gamma$  compact and  $E$  is a Hausdorff space then:

- (i) The law of action  $\varphi: E \times \Gamma \rightarrow E$  is a closed map.
- (ii)  $E/\Gamma$  is Hausdorff.
- (iii) The map  $\pi: E \rightarrow E/\Gamma$  is proper.
- (iv)  $E$  is compact if and only if  $E/\Gamma$  is compact.

**3.5 Definition [Majeed\*\*,2018]:**

A topological groupoid  $(G,B)$  is called source proper groupoid "sp- groupoid" if:

- (i) The source map  $\alpha: G \rightarrow G$  is a proper.
- (ii) The basic space  $B$  is proper.
- (iii) The unit map  $w: B \rightarrow G$  is proper.

**3.7 Proposition [Majeed\*\*,2018]:**

Let  $(G,B)$  be an sp- groupoid then the map  $\delta_x: G_x \times G_x \rightarrow G$ , defined by  $\delta_x(g_1, g_2) = y(g_1, \sigma(g_2))$  is a proper map, for every  $x \in B$ .

**3.8 Example of  $\Gamma - \text{space}$  [IIIman,2002],[Bredon,1972]:**

Let  $\Gamma = R$  the real additive topological group and  $E=R$  then  $R$  is  $R$ -space by  $: R \times R \rightarrow R, \phi(r, t) = r + t$  for all  $(r, t) \in R \times R$ .

**3.9 Definition [Majeed\*,2018]:**

A  $\Gamma - \text{space}$   $E$  is called a source proper group space ((sp  $\Gamma - \text{space}$ )) if:

- (i) The action groupoid  $(E \times \Gamma, E)$  is an Sp-groupoid.
- (ii)  $E$  is a free  $\Gamma - \text{space}$ .

**3.10 Example [Majeed\*,2018]:**

R is R-space (Example(2.8)) and R is not SpR-space since action groupoid  $(R \times R, R)$  is not Sp-groupoid.

**3.11 Proposition:**

Let  $E_i$  be an Sp  $\Gamma$  –space ,  $i = 1,2,3$  then  $E_1 \times E_2 \times E_3$  is an Sp  $\Gamma$  –space.

**Proof:**

Define  $\Psi((z_1, z_2, z_3), r) = (\Phi_1(z_1, r), \Phi_2(z_2, r), \Phi_3(z_3, r))$  for every  $(z_1, z_2, z_3) \in E_1 \times E_2 \times E_3$  and  $r \in \Gamma$ , which is continuous , where  $\Phi_1$  is a law of action of  $\Gamma$  , on  $E_i, i = 1,2,3$ .

Now if  $\Psi((z_1, z_2, z_3), r) = (z_1, z_2, z_3)$  then  $r = e$  since  $E_i$  is a free  $\Gamma$  –space,  $i = 1,2,3$ .

Hence  $E_1 \times E_2 \times E_3$  is a free  $\Gamma$  –space and the action groupoid  $((E_1 \times E_2 \times E_3) \times \Gamma, E_1 \times E_2 \times E_3)$  is an Sp- groupoid

since  $E_1 \times E_2 \times E_3$  is a Hansdorff ( $E_1, E_2$  and  $E_3$  are Hansdorff) and the source map.

$\alpha: (E_1 \times E_2 \times E_3) \times \Gamma \rightarrow E_1 \times E_2 \times E_3, \alpha((z_1, z_2, z_3), r) = (z_1, z_2, z_3) = (\alpha_1(z_1, r), \alpha_2(z_2, r), \alpha_3(z_3, r))$  is proper by using the following commutative diagram in T:

$$\begin{array}{ccc}
 (E_1 \times \Gamma) \times (E_2 \times \Gamma) \times (E_3 \times \Gamma) & \longrightarrow & E_1 \times E_2 \times E_3 \\
 \downarrow f & & \downarrow I_{E_1 \times E_2 \times E_3} \\
 (E_1 \times E_2 \times E_3) \times \Gamma & \xrightarrow{\alpha} & E_1 \times E_2 \times E_3
 \end{array}$$

and (proposition (1.8 , ii) and proposition (1.9, ii).

Where f is the map defined by:

$$f((z_1, r), (z_2, r), (z_3, r)) = ((z_1, z_2, z_3), r)$$

which is a surjective continuous since

$$\begin{aligned}
 f &= : (E_1 \times \Gamma) \times (E_2 \times \Gamma) \times (E_3 \times \Gamma) \xrightarrow{\cong} (E_1 \times E_2 \times E_3) \times \Gamma \times \Gamma \times \Gamma \xrightarrow{Pr_{123}} (E_1 \times E_2 \times E_3) \times \Gamma \\
 &((z_1, r), (z_2, r), (z_3, r)) \rightarrow ((z_1, z_2, z_3), (r_1, r_2, r_3)) \rightarrow ((z_1, z_2, z_3), r) .
 \end{aligned}$$

Therefore  $E_1 \times E_2 \times E_3$  is an Sp-space.

**3.12 Remark:**

As a consequence of (Proposition (2.11)) we have  $E \times E \times E$  is Sp-space when E is an Sp-space.

**3.13 Proposition [Mackenzie,1987, proposition 2.12, page22]:**

Let  $\Gamma(E, \pi, B)$  be a cartan principle bundle then  $E \times E/\Gamma$  is a topological groupoid of base B.

The pair  $(E \times E/\Gamma, B)$  is called Ehresman groupoid.  $E \times E/\Gamma$  has the structure of groupoid as follows:

$$\alpha([(z, \acute{z})]) = \pi(\acute{z}), \beta([(z, \acute{z})]) = \pi(z), w(b) = [(z, z)]$$

where  $z \in \pi^{-1}(b)$ , the law of partial composition is given by:

$$[(z, \acute{z})][(\bar{z}, \acute{\acute{z}})] = [(z, \acute{\acute{z}})] \text{ if and only if } \pi(\acute{z}) = \pi(\bar{z}) \text{ and the inverse of } [(z, \acute{z})] \text{ is } [(\acute{z}, z)].$$

In the previous Example the groupoid  $S \cup (2) \times S \cup (2)/z_2$  can be naturally identity with  $SO(4)$ .

Not that , if E be an Sp-space the E is a free  $\Gamma$ -space and then for every  $(z, \acute{z}) \in E \times E$  ((the fiber product of  $\pi: E \rightarrow B = E/\Gamma^B$  by itself)) there is a unique  $r \in \Gamma$  such that  $\acute{z} = \varphi(z, r)$ .

So we have a map  $\delta^*$  from  $E \times E$  to  $\Gamma$  defined by  $\delta^*(z, \acute{z}) = r$ . In general the map  $\delta^*$  is not continuous when E is a free  $\Gamma$ -space [6].

**3.14 Proposition [Majeed\*\*,2018]:**

Let E be an Sp $\Gamma$  –space then:

- (i) The map  $\delta^*: E_B \times E \rightarrow \Gamma$  is continuous.
- (ii)  $E_B \times E$  is a closed subgroupoid of Des'cartes groupoid  $E \times E$ .
- (iii) The orbit maps are proper maps for every  $z \in E$ .

**3.15 Proposition :**

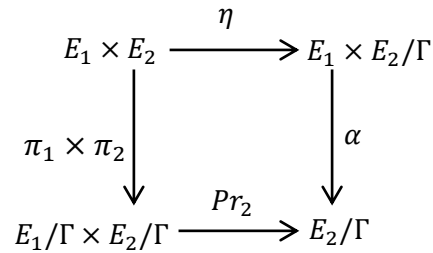
Let  $E_i$  be an Sp $\Gamma$  –space,  $i = 1,2$  then  $E_1 \times E_2$  is an Sp $\Gamma$  –space and the set of all orbits  $G = E_1 \times E_2/\Gamma$  is an Sp –groupoid of base  $= E_i/\Gamma, i = 1,2$  with identification topology associated to morphism  $\eta = E_1 \times E_2 \rightarrow E_1 \times E_2/\Gamma, \eta(z, \acute{z}) = [(z, \acute{z})]$  .

**Proof:**

$E_1 \times E_2$  is an Sp $\Gamma$  –space (Proposition(1.3)) by  $\psi: (E_1 \times E_2) \times \Gamma \rightarrow E_1 \times E_2$  , defined by  $\psi((z, \acute{z}), r) = (\varphi(\acute{z}, r))$

Where  $\varphi$  is the action of  $\Gamma$  on  $E_i$ ,  $i = 1,2$ . To show that  $(G = E_1 \times E_2/\Gamma, B = E_i/\Gamma, i = 1,2)$  is an  $Sp$  – groupoid.

(i)  $(G = E_1 \times E_2/\Gamma, B = E_i/\Gamma, i = 1,2)$  is a topological groupoid since the unit map  $w$ , the inversion map  $\sigma$  are both continuous (Proposition(2.13)) and the law of partial composition  $\gamma$  is continuous (Proposition(2.14,i)) and then (Proposition(2.13)) and source map  $\pi_1 \times \pi_2: E_1 \times E_2 \rightarrow E_1/\Gamma \times E_2/\Gamma$  is constant on the fibers of the identification map  $\eta$ , hence there is unique morphism  $\tau = \tau: G \rightarrow B_1 \times B_2$  in  $T$  for which the following diagram is commutative in  $T$ :



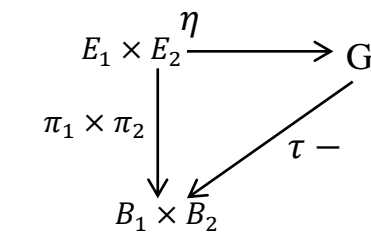
In which  $\eta$  and  $\pi_1 \times \pi_2$  are closed (Theorem(2.4, iii)) and proposition (1.9,i) and  $Pr_2$  is closed (proposition(1.7)). Hence  $\alpha$  is a closed map. Therefore  $\alpha$  is a proper map and  $(E_1 \times E_2/\Gamma, E_2/\Gamma)$  is a groupoid .

**3.16 Corollary:**

Let  $(G, B)$  be an  $Sp$  – groupoid then the  $\alpha$  – fiber space  $\otimes_{i=1}^n G_{x_i}$  and  $\otimes_{i=1}^n G_{y_i}$  are isomorphic group spaces for any  $g_i \in G_{x_i}, i = 1,2, \dots, n$  .

**Proof:**

$\otimes_{i=1}^n G_{x_i}$  are  $Sp_x G_x$  –space and  $\otimes_{i=1}^n G_{y_i}$  are  $Sp_y G_y$ –space (proposition(1.7)).



by the universal property of identification map so  $\alpha: G = E_1 \times E_2/\Gamma \xrightarrow{\tau^{-1}} B_1 \times B_2 \xrightarrow{pr_1} B_1$  is continuous and the target map is continuous since  $\beta = \alpha \circ \sigma$ .

(ii) the base space  $B = E_i/\Gamma, i = 1,2$  is Hausdorff. (theorem(2.4,i))

(iii) To prove the source map  $\alpha: E_1 \times E_2/\Gamma \rightarrow E_i/\Gamma, i = 1,2$ ,  $\alpha([(z, z)]) = \pi_2(z)$  is proper.

The map  $\Gamma \rightarrow \{(z, z_0)\} \times \Gamma \xrightarrow{\Psi^*} ((z, z_0), \Gamma)$  is continuous and then each orbit  $\psi((z, z_0), \Gamma)$  is compact ( $\Gamma$  is compact) where  $\psi^* = \psi|_{\{(z, z_0)\} \times \Gamma}$ .

But  $\alpha$  –fiber space  $G_{\pi(z_0)} = \alpha^{-1}(\pi(z_0))$  is a closed subspace of  $\psi((z, z_0), \Gamma)$  since  $E_i/\Gamma, i = 1,2$  is Hausdorff. (theorem(2.4,ii)).

Thus  $\alpha$  –fiber space  $G_{\pi(z_0)}$  is compact for every  $z_0 \in E_i, i=1,2$ .

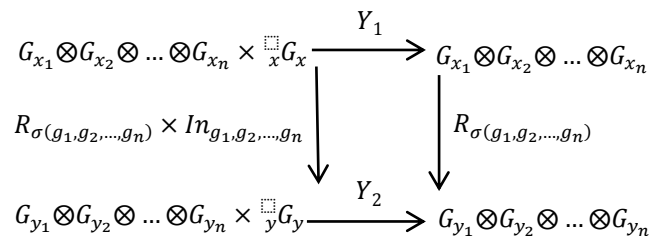
Hence the fiber of  $\alpha$  are compact.

Now, to show that  $\alpha$  is a closed map, the map  $f_{(z_0)}: E_i \rightarrow G_{\pi(z_0)}, i = 1,2$ , defined by  $f_{(z_0)}(z) = [(z, z_0)]$  is homeomorphism[8].

Thus  $E_1, E_2$  are compact ( $G_{\pi(z_0)}$  are compact) and  $E_i/\Gamma, i = 1,2$  is compact.

Now consider the following commutative diagram in  $T$ :

$\otimes_{i=1}^n G_{x_i}$  homeomorphic to  $\otimes_{i=1}^n G_{y_i}$  by  $R_{\sigma(g_1, g_2, \dots, g_n)} = G_{x_1} \otimes G_{x_2} \otimes \dots \otimes G_{x_n} \rightarrow G_{y_1} \otimes G_{y_2} \otimes \dots \otimes G_{y_n}$ ,  $R_{\sigma(g_1, g_2, \dots, g_n)}(h_1, h_2, \dots, h_n) = Y(h_1, h_2, \dots, h_n, \sigma(g_1, g_2, \dots, g_n))$  and vertex group  $_x G_x$  isomorphic to vertex group  $_y G_y$  by inner automorphism  $In_{g_1, g_2, \dots, g_n}(h_1, h_2, \dots, h_n) = Y(g_1, g_2, \dots, g_n, Y(h_1, h_2, \dots, h_n, \sigma(g_1, g_2, \dots, g_n)))$  and the following diagram is commutative in  $T$ :



Where  $Y_1 = Y \setminus_{G_{x_1} \otimes G_{x_2} \otimes \dots \otimes G_{x_n} \times G_x}$  and  $Y_2 = Y \setminus_{G_{y_1} \otimes G_{y_2} \otimes \dots \otimes G_y \times G_y}$ .

Hence the pair  $(R_{\sigma(g_1, g_2, \dots, g_n)} \times In_{g_1, g_2, \dots, g_n})$  represent an isomorphic of group spaces.

**3.17 Remark:**

As a consequence of corollary(2.15) we have an isomorphic class of source proper group spaces

$$[Sp_x G_x - \text{spac} \otimes_{i=1}^n G_{x_i}] X \in B.$$

**3.18 Proposition:**

Let  $E_1 \times E_2$  be an  $Sp\Gamma$  -space and  $\dot{E}_1 \times \dot{E}_2$  be equivariant space of  $E_1 \times E_2$ , if  $\bar{E}_1 \times \bar{E}_2$  is a Hansdorff space and

$g: \bar{E}_1 \times \bar{E}_2 \rightarrow \dot{E}_1 \times \dot{E}_2$  is a continuous map then  $(\bar{E}_1 \times \bar{E}_2) \times_{\dot{E}_1 \times \dot{E}_2} (E_1 \times E_2)$  is the fiber product of equivariant maps  $f$  and  $g$  over  $\dot{E}_1 \times \dot{E}_2$ .

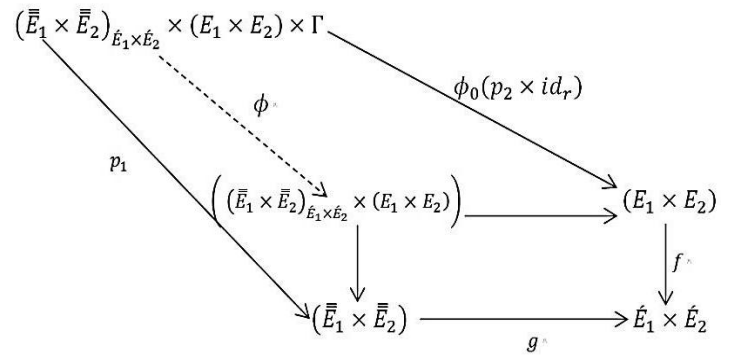
**Proof:** Define

$$\phi: \left( (\bar{E}_1 \times \bar{E}_2) \times (E_1 \times E_2) \times \Gamma \rightarrow (\bar{E}_1 \times \bar{E}_2) \times_{\dot{E}_1 \times \dot{E}_2} (E_1 \times E_2) \right)$$

$$\phi: ((\bar{z}_1 \times \bar{z}_2), (z_1 \times z_2), r) = (\bar{z}_1 \times \bar{z}_2, \varphi((z_1 \times z_2), r))$$

where  $\varphi$  is a law of action of  $\Gamma$  on  $E_1 \times E_2$ .  $\phi$  is a continuous action of  $\Gamma$  on  $(\bar{E}_1 \times \bar{E}_2) \times_{\dot{E}_1 \times \dot{E}_2} (E_1 \times E_2)$ , since:

- (i)  $\phi((\bar{z}_1 \times \bar{z}_2), (z_1 \times z_2), e) = (\bar{z}_1 \times \bar{z}_2, \varphi((z_1 \times z_2), e)) = ((\bar{z}_1 \times \bar{z}_2), (z_1 \times z_2))$  since  $((z_1 \times z_2), e) = (z_1 \times z_2)$ , for every  $((\bar{z}_1 \times \bar{z}_2), (z_1 \times z_2)) \in (\bar{E}_1 \times \bar{E}_2) \times_{\dot{E}_1 \times \dot{E}_2} (E_1 \times E_2)$ .
- (ii)  $\phi((\bar{z}_1 \times \bar{z}_2), (z_1 \times z_2), M(r_1 \times r_2)) = ((z_1 \times z_2), \varphi((z_1 \times z_2), M(r_1 \times r_2))) = ((\bar{z}_1 \times \bar{z}_2), \varphi(\varphi((z_1 \times z_2), r_1), r_2)) = \phi(\phi(\bar{z}_1 \times \bar{z}_2), (z_1 \times z_2), r_1), r_2)$ .
- (iii) If  $\phi((\bar{z}_1 \times \bar{z}_2), (z_1 \times z_2), r_1) = ((\bar{z}_1 \times \bar{z}_2), (z_1 \times z_2)) \Rightarrow \varphi(z_1 \times z_2, r) = (z_1 \times z_2) \Rightarrow r = e$ , since  $E_1 \times E_2$  is free  $\Gamma$  - space.
- (iv) Consider the following diagram:



In which

$f \circ \phi - (p_2 \times id_r)((\bar{z}_1 \times \bar{z}_2), (z_1 \times z_2), r) = g \circ p_1((\bar{z}_1 \times \bar{z}_2), (z_1 \times z_2), r)$ . Hence there exist a unique morphism

$\theta = \phi: (\bar{E}_1 \times \bar{E}_2) \times (E_1 \times E_2) \times \Gamma \rightarrow (\bar{E}_1 \times \bar{E}_2) \times_{\dot{E}_1 \times \dot{E}_2} (E_1 \times E_2) = ((\bar{z}_1 \times \bar{z}_2), \varphi(z_1 \times z_2), r)$  making the whole diagram commutative in  $\Gamma$  by the universal property of fiber product. Hence

$(\bar{E}_1 \times \bar{E}_2) \times_{\dot{E}_1 \times \dot{E}_2} (E_1 \times E_2)$  is a free  $\Gamma$  -space.

To show that the action groupoid

$(\bar{E}_1 \times \bar{E}_2) \times_{\dot{E}_1 \times \dot{E}_2} (E_1 \times E_2) \times \Gamma$  is an  $Sp$  - groupoid.

$(\bar{E}_1 \times \bar{E}_2)$  is a Hansdorff space and  $(E_1 \times E_2)$  is a Hansdorff space (Definition (2.9)).

Hence  $(\bar{E}_1 \times \bar{E}_2) \times_{\dot{E}_1 \times \dot{E}_2} (E_1 \times E_2)$  is a

Hansdorff space (subspace) of Hansdorff  $(\bar{E}_1 \times$

$\bar{E}_2) \times_{\dot{E}_1 \times \dot{E}_2} (E_1 \times E_2)$ . The fibers

$\alpha^{-1}((z_1 \times z_2)) = \{(z_1 \times z_2)\} \times \Gamma$  of the source map of the action groupoid  $((E_1 \times E_2) \times \Gamma, (E_1 \times E_2))$  are compact since  $(E_1 \times E_2)$  is an  $Sp\Gamma$  -space, but  $\{(z_1 \times z_2)\} \times \Gamma \cong \Gamma$ .

Hence the source map of the action groupoid

$(\bar{E}_1 \times \bar{E}_2) \times_{\dot{E}_1 \times \dot{E}_2} (E_1 \times E_2) \times \Gamma, (\bar{E}_1 \times \bar{E}_2) \times_{\dot{E}_1 \times \dot{E}_2} (E_1 \times E_2)$  is proper (proposition(1.7)).

**4- Submersive group spaces.**

**4.1 Definition [Mackenzie,1987]:**

A topological submersion map is a continuous map

$f: X \rightarrow Y$ , such that for every  $x_0 \in X$ , there is an open neighborhood  $U$  of  $f(x_0)$  in  $Y$  and a continuous right inverse

$g: U \rightarrow X$ , to  $f$ , such that  $g(f(x_0)) = x_0$ .

Now, we introduce a special class of source proper groupoid which is called submersive groupoid and define it as follows:

**4.2 Definition [Majeed\*\*,2018]:**

A transitive  $Sp$  – groupoid  $(G, B)$  is called submersive groupoid (SSp- groupoid), if the Map  $\beta_x: G_x \rightarrow B$  is submersion for every  $x \in B$ .

**4.3 Example [Majeed\*,2018]:**

Every compact transitive topological groupoid  $G$  on a discrete space  $B$  is an SSp- groupoid. Since for every  $g \in G_x$  then

$U = \{B_x(g)\}$  is an open neighborhood in  $B$  of  $B_x(g)$  and the constant map  $V: U \rightarrow \{B_x(g) = g\}$  is a continuous right inverse to

$B_x: G_x \rightarrow B$ .

Now, we introduce a special class of source proper group space called submersive group space and define it as follows:

**4.4 Definition [Majeed\*,2018]:**

An  $Sp$  –space  $E$  is called submersive group space " SSp- space" if the map  $\pi: E \rightarrow E/\Gamma$  is submersion.

**4.5 Proposition:**

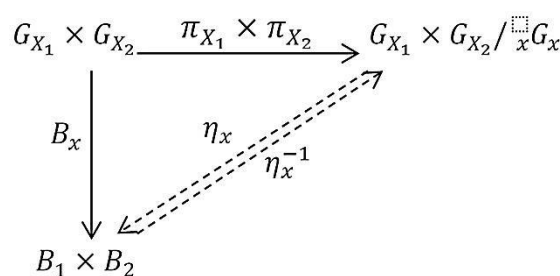
Let  $(G, B)$  be an SSp- groupoid then  $G_{X_1} \times G_{X_2}$

is an  $SSp_{xG_x}$  – space, for every  $x \in B$ .

**Proof :**

$G_{X_1}$  and  $G_{X_2}$  are  $Sp_{xG_x}$ -space  $\Rightarrow G_{X_1} \times G_{X_2}$  is  $Sp_{xG_x}$ -space, (proposition(1.3)).

To show that the map  $\pi_x: G_{X_1} \times G_{X_2} \rightarrow G_{X_1} \times G_{X_2}/_xG_x$  is submersion, for all  $x \in B$ . The maps  $\pi_{X_1} \times \pi_{X_2}: G_{X_1} \times G_{X_2} \rightarrow G_{X_1} \times G_{X_2}/_xG_x$  and  $B_x: G_{X_1} \times G_{X_2} \rightarrow B_1 \times B_2$  are both identification maps ( $B_x$  is surjective proper map, proposition (2.6, ii) and (1.7)) and constant on the fiber of each others. Hence the dotted arrows in the following diagram:



exist and are unique in  $T$  by universal property of identification and the map  $\eta$  is given by

$\eta_x(\pi_{X_1} \times \pi_{X_2}(g_1, g_2)) = B_x(g_1, g_2)$ . Now, to show that the map  $\pi_{X_1} \times \pi_{X_2}: G_{X_1} \times G_{X_2} \rightarrow G_{X_1} \times G_{X_2}/_xG_x$  is submersion.

Let  $(g_1, g_2) \in G_{X_1} \times G_{X_2}$ ,  $\pi_{X_1} \times \pi_{X_2}(g_1, g_2) \in G_{X_1} \times G_{X_2}/_xG_x$ ,  $\eta_x(\pi_{X_1} \times \pi_{X_2}(g_1, g_2)) = B_x(g_1, g_2)$  then there is an open neighborhood  $U_{(g_1, g_2)}$  of  $B_x(g_1, g_2)$  in  $B_1 \times B_2$  and a continuous right inverse  $V: U \rightarrow G_{X_1} \times G_{X_2}$  to  $B_x: G_{X_1} \times G_{X_2} \rightarrow B_1 \times B_2$  such that  $V \circ B_x(g_1, g_2) = (g_1, g_2)$ , ( $B_x$  is submersion  $(G, B)$  is SSp – groupoid). Now, define

$\hat{V}_{(g_1, g_2)}: \eta_x^{-1}(U_{(g_1, g_2)}) \xrightarrow{\eta_x} (U_{(g_1, g_2)}) \xrightarrow{V} G_{X_1} \times G_{X_2}$  by  $\hat{V}_{(g_1, g_2)}(r) = V \circ \eta_x(r)$  where  $\eta_x^{-1}(U_{(g_1, g_2)})$  is an open neighborhood of  $\pi_{X_1} \times \pi_{X_2}(g_1, g_2)$  in  $G_{X_1} \times G_{X_2}$ .  $\hat{V}(g_1, g_2)$  is continuous and

$$(\pi_{X_1} \times \pi_{X_2}) \circ \hat{V}_{(g_1, g_2)}(r) = (\pi_{X_1} \times \pi_{X_2}) \circ V \circ \eta_x(r) =$$

$$\eta_x^{-1} \circ \eta_x \circ (\pi_{X_1} \times \pi_{X_2}) \circ V \circ \eta_x(r) = \eta_x^{-1} \circ B_x \circ V \circ (\pi_{X_1} \times \pi_{X_2})(r) = \eta_x^{-1} \circ \eta_x(r) = r \quad \text{and}$$

$$\hat{V}_{(g_1, g_2)}(\pi_{X_1} \times \pi_{X_2}(g_1, g_2)) = V \circ \eta_x(\pi_{X_1} \times \pi_{X_2}(g_1, g_2)) = V \circ B_x(g_1, g_2) = g.$$

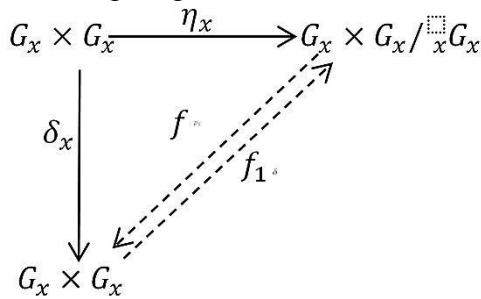
**4.6 Proposition:**

Let  $(G_{X_1} \times G_{X_2}, B_1 \times B_2)$  be transitive Sp- groupoid then Ehressman groupoid  $(G_{X_1} \times G_{X_2}/_xG_x, G_x/_xG_x)$  isomorphic to  $(G_{X_1} \times G_{X_2}, B_1 \times B_2)$  in  $T_G$ .

**Proof:**

The map  $\delta_x: G_x \times G_x \rightarrow G_1 \times G_2$  is surjective proper (proposition(2.7)). then the maps  $\delta_x: G_x \times G_x \rightarrow G_1 \times G_2$  and

$\eta_x: G_x \times G_x \rightarrow G_x \times G_x / {}_xG_x$  are both identification maps and constant on the fiber of each other. Hence the dotted arrows in the following diagram:



exist and are unique in T by universal property of identification map, and map f is given by :

$f([(g_1, g_2)]) = (Y(g_1, \sigma(g_2)), g_1)$  becomes homeomorphism.

Clearly  $(f, \eta_x)$  is an isomorphism of topological groupoid where  $\eta_x$  is the map given by  $\eta_x(\pi_x(g)) = B_x(g)$  where  $\pi_x: G_x \rightarrow G_x / {}_xG_x$  (proposition (3.5)).

**4.7 proposition:**

Let  $E \times E$  be an SSp- space then Ehresman groupoid

$((E \times E) \times E/\Gamma, E/\Gamma)$  is submersive groupoid.

**Proof:**

Let  $G = (E \times E) \times E/\Gamma, B = E/\Gamma, (G, B)$  is transitive Sp- groupoid (proposition (2.15) to show that the map  $B_x: G_x \rightarrow B$  is submersion .

Let  $g = [(z, \acute{z}), \bar{z}] \in f_x$  , then  $B_x(g) = \pi(z, \acute{z})$  in B and continuous right inverse  $V^*: U \rightarrow E \times E$  such that

$V^*(\pi(z, \acute{z})) = (z, \acute{z})(\pi: E \times E \rightarrow B = E/\Gamma)$  is submersion, Definition (2.4).

Define  $\acute{V}^*: U \rightarrow G_x$  by  $\acute{V}^*(y) = [(V^*(y), \acute{z}), \bar{z}]$  for every  $y \in U$ .

$B_x \circ \acute{V}^*(y) = B_x([(V^*(y), \acute{z}), \bar{z}]) = \pi \circ V^*(y) = y = I_U$  for every

$$\begin{aligned}
 y \in U \text{ and } \acute{V}^*(y) \circ B_x(g) &= \acute{V}^*(B_x([(z, \acute{z}), \bar{z}])) \\
 &= \acute{V}^*(\pi(z, \acute{z})) \\
 &= [(\acute{V}^*(\pi(z, \acute{z})), \bar{z})] \\
 &= [(z, \acute{z}), \bar{z}] = g
 \end{aligned}$$

**4.8 Corollary:**

Let  $(G, B)$  be an Ssp- groupoid then Ehresman groupoid

$(G_x \times G_x / {}_xG_x, G_x / {}_xG_x)$  is an Ssp- groupoid for every  $x \in B$ .

**Proof:**

Let  $G = G_x \times G_x / {}_xG_x, B = G_x / {}_xG_x, (G, B)$  is a transitive Sp- groupoid (proposition(2.15)) to show that the map  $B_x: G_x \rightarrow B$  is submersion.

Let  $g = [(z, \acute{z})] \in G_x$  , then  $B_x(g) = \pi(z)$  and there exist an open neighborhood  $U$  of  $\pi(z)$  in  $B$  and a continuous right inverse  $V^*: U \rightarrow G_x$  to  $\pi$  such that  $V^* \circ \pi(z) = z$  ( $\pi: G_x / {}_xG_x$ ) is submersion, Definition (1.3).

Define  $\acute{V}^*: U \rightarrow G_x$  by  $\acute{V}^*(y) = [(V^*(y), \acute{z})]$  for every  $y \in U$ .

$$\begin{aligned}
 B_x \circ \acute{V}^*(y) &= B_x([(V^*(y), z)]) \\
 &= \pi \circ V^*(y) = y = I_U
 \end{aligned}$$

for every  $y \in U$  and  $\acute{V}^*(y) \circ B_x(g) = \acute{V}^*(B_x([(z, \acute{z})])) = \acute{V}^*(\pi(z)) = [(\acute{V}^*(\pi(z)), \acute{z})] = [(z, \acute{z})] = g$ .

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