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RESEARCH PAPER

Generalized Taylor Matrix Method for Solving Multi-Higher Nonlinear Integro-Fractional Differential Equations of Fredholm Type

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A B S T R A C T:

In this study, generalized Taylor expansion approach formula is developed for solving approximately a Fredholm-Hammerstein type of multi-higher order nonlinear integro- fractional differential equations with variable coefficients under given mixed conditions. The fractional derivative is described in the Caputo sense. Using the collocation points, this new technique depends mainly on transform the nonlinear equation and conditions into the matrix equations which leads to solve a system of nonlinear algebraic equations with unknown generalized Taylor coefficients. A best algorithm for solving our equation numerically by applying this process has been developed in order to express these solution, programs are written in MatLab. In addition, the truth and reliability of this method is tested by several illustrative numerical examples are presented to show effectiveness and accuracy of this algorithm.

KEY WORDS:Nonlinear Integro-Fractional Differential Equation, Generalized Taylor's Method, Multinomial Theorem, Collocation Points, Caputo Fractional Derivative.
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1. INTRODUCTION

In this work we define the multi-higher order nonlinear integro-fractional differential equations (IFDE) of Fredholm-Hammerstein (F-H) type with variable coefficients and the fractional derivative in Caputo sense of order $m\alpha$ for $0 < \alpha \le 1, m \in \mathbb{Z}^+$ and $\beta_j \in \mathbb{R}^+, \forall j$:

$$ {}_a^C D_t^{n\alpha} y(t) + \sum_{i=1}^{n-1} \mathcal{P}_i(t)\, {}_a^C D_t^{(n-i)\alpha} y(t) + \mathcal{P}_n(t) y(t) = f(t) + \sum_{j=0}^{m} \lambda_j \int_a^b \mathcal{K}_j(t,s) \mathcal{H}_j\left(s, {}_a^C D_s^{\beta_j} y(s)\right) ds \quad \dots (1) $$

with the mixed conditions:

$$ Z_k\left(y^{(0)}(c_\xi), y^{(1)}(c_\xi), \cdots, y^{(k)}(c_\xi)\right) = C_k\ ; \quad C_\xi \in [a,b]\ \forall\ \xi $$

For all $k = 0, 1, \cdots, \mu - 1$; s.t. μ is define by:

$$ \mu = \max\{\lceil \alpha_n \rceil, \lceil \beta_j \rceil : j = 0,1,\dots,m\} $$

where Z_k are linear combination of $y^{(0)}(c_\xi), y^{(1)}(c_\xi), \cdots, y^{(\mu-1)}(c_\xi)$. Further, $y(t)$ is the unknown function which is the solution of equation (1) the functions $\mathcal{K}_j: S \to \mathbb{R}$ with $(S = \{(t,s): a \le s \le t \le b\}; \mathcal{H}_j \in C([a,b] \times \mathbb{R}, \mathbb{R})$; $j = 0, 1, \cdots, m$ and $f, p_i: [a,b] \to \mathbb{R}$; $i = 0,1,2,\cdots,n$ are all continuous functions. In addition that the fractional orders $\alpha_i, \beta_j \in \mathbb{R}^+$, for all $(i,j \ne 0)$, $m_{\alpha_i} - 1 < \alpha_i \le m_{\alpha_i}$ and $m_{\beta_j} - 1 < \beta_j \le m_{\beta_j}$, $m_{\alpha_i} = \lceil \alpha_i \rceil$ and $m_{\beta_j} = \lceil \beta_j \rceil$ for all $i = 1,2,\dots,n$ and $j = 1,2,\dots,m$, with property that $\alpha_n > \alpha_{n-1} > \cdots > \alpha_2 > \alpha_1 > \alpha_0 = 0$.

The Taylor expansion method is a powerful technique to approximate the value of the unknown function at an arbitrary position, it is a series expansion around a given point. As well as,

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this method is one of the numerical procedures for computing an approximated solution to solve different types of equations linear and nonlinear forms. For integral equations, the Taylor polynomial method has been suggested to find the solution approximately. First of all, Kanwal and Liu (Kanwal *et al.*, 1989) used it. This method has been extended to VIEs by Sezer (Sezer, 1994) and Bellour with Rawashdeh (Bellour *et al.*, 2010) applied it to find an approximate solution for first kind integral equations. Yalcinbas (Yalcinbas, 2002) and Mahmoudi (Mahmoudi, 2005) used approach to solve nonlinear V-F integral equations, while for nonlinear V-F integro-Differential equations, Maleknejad and Mahmoudi (Muleknejad *et al.*, 2003) and Darania with Ivaz (Darania *et al.,* 2008) are all used a similar approach. Moreover, Taylor polynomial method have been presented in many works in papers (Darania *et al.,* 2006; Dascioglu *et al.,* 2007; Gulsu *et al.,* 2006; Huang *et al.,* 2010; J. Rashidinia *et al.*, 2012) in different procedures. In recent years many author have presented the same procedures to solve fractional integro-Differential equations (Huang *et al.*, 2011; Saleh *et al.*, 2013). Also, a new generalized Taylor's formula that involves Caputo-fractional derivatives was suggested to solve problems in terms of fractional integro-Differential equations (Shazad *et al.*, 2011).

Here, a simple and important previous technique depends mainly on the generalized Taylor expansion approach, are developed and applied to solve such a problem. This new proposed scheme transform, a multi-higher order nonlinear FIDE of F-H type with variable coefficients and given mixed conditions into a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown Taylor expansion coefficients of the solution function. Finally, applying Newton's method to solve this nonlinear algebraic equation for determine the unknown generalized Taylor coefficients, the truncated generalized Taylor series approach is obtained. A best algorithm and good programs are written in general cases, this method is tested by several illustrative numerical examples to show effectiveness and accuracy of it.

2. BASIC DEFINITIONS; GENERALIZED FORMULAS:

For completeness, here we present the necessary definitions and some notations of fractional calculus theory, which are used throughout this paper. For more details, see all: (Anatoly *et al.*, 2006; podldubny, 1999; Kelth *et al.*, 1974; Kenneth *et al.*, 1993; Shazad, 2009).

Definition 2.1:

A real valued function y defined on $[a,b]$ be in the space $C_\epsilon[a,b]$, ϵ-any real number, if there exists a real number $k>\epsilon$, such that $y(t)=(t-a)^k\overline{y}(t)$, where $\overline{y}\in C[a,b]$, and it is said to be in the space $C_\epsilon^n[a,b]$ if and only if $y^{(n)}\in C_\epsilon[a,b]$, n-positive integer number with zero.

Definition 2.2:

Let $u\in C_\epsilon[a,b], \epsilon\geq -1$ with any positive arbitrary real number α. Then the Riemann-Liouville fractional integral operator ${}_aJ_t^\alpha$ of order α of a function y, is defined as:

$$ {}_aJ_t^\alpha y(t)=\begin{cases}\int_a^t \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}y(\xi)d\xi, & \alpha>0\\ y(t) \quad \text{whenever} & \alpha=0\end{cases} $$

Definition 2.3:

Let $\alpha\geq 0$, and $m=\lceil\alpha\rceil$. the Riemann-Liouville fractional derivative operator ${}_a^RD_t^\alpha$, of order α and $y\in C_{-1}^m[a,b]$ and defined as:

$$ {}_a^RD_t^\alpha y(t) =\begin{cases} D_t^m[{}_aJ_t^{m-\alpha}y(t)], & \alpha>0\\ y(t) \quad \text{whenever} & \alpha=0\\ y^{(m)}(t), \text{If } \alpha=m(\in\mathbb{N}) \text{ and } y\in C^m[a,b] & \end{cases} $$

Definition 2.4:

The Caputo fractional derivative operator ${}_a^CD_t^\alpha$ of order $\alpha\in\mathbb{R}^+$ of a function $y\in C_{-1}^m[a,b]$ and $m-1<\alpha\leq m,\ m\in\mathbb{N}$ is defined as:

$$ {}_a^CD_t^\alpha y(t) =\begin{cases} {}_aJ_t^{m-\alpha}[D_t^m y(t)], & \alpha>0\\ y(t) \quad \text{whenever} & \alpha=0\\ y^{(m)}(t), \text{If } \alpha=m(\in\mathbb{N}) \text{ and } y\in C^m[a,b] & \end{cases} $$

Hence, we have the following properties:

- For $\alpha\geq 0$ and $\beta>0$, then:
$$ {}_aJ_t^\alpha(t-a)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}. $$
- For all $\alpha\geq 0,\ \beta\geq 0$ and $y(t)\in C_\epsilon[a,b],\ \epsilon\geq -1$, then:
$$ {}_aJ_t^\alpha\, {}_aJ_t^\beta y(t)={}_aJ_t^\beta\, {}_aJ_t^\alpha y(t)={}_aJ_t^{\alpha+\beta}y(t) $$
- ${}_a^RD_t^\alpha\mathcal{C}=\mathcal{C}\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$ and ${}_a^CD_t^\alpha\mathcal{C}=0$; $\mathcal{C}$ is any constant; $(\alpha\geq 0,\alpha\notin\mathbb{N})$
- ${}_a^RD_t^\alpha y(t)=D_t^m\,{}_aJ_t^{m-\alpha}y(t)\neq {}_aJ_t^{m-\alpha}D_t^m y(t)={}_a^CD_t^\alpha y(t)\ ; m=\lceil\alpha\rceil$.

- Assume that $y \in C_{-1}^{m}[a,b]$; $\alpha \geq 0, \alpha \notin \mathbb{N}$ and $m = \lceil \alpha \rceil$ then ${}_a^C D_t^\alpha y(t)$ is continuous on $[a,b]$, and $[{}_a^C D_t^\alpha y(t)\,]_{t=a} = 0$.
- Let $\alpha \geq 0$, $m = \lceil \alpha \rceil$ and $y \in C^m[a,b]$, then, the relation between the Caputo derivative and R-L integral are formed:

$$ {}_a^C D_t^\alpha [{}_a J_t^\alpha y(t)] = y(t) \ ; \quad a \leq t \leq b $$

$$ {}_a J_t^\alpha \ [{}_a^C D_t^\alpha y(t)] = y(t) - \sum_{k=0}^{m-1} \frac{y^{(k)}(a)}{k!}(t-a)^k $$

- Let $\alpha \geq 0; m = \lceil \alpha \rceil$; for $y(t) = (t-a)^\beta$ for some $\beta \geq 0$. Then:

$$ {}_a^C D_t^\alpha y(t) = \begin{cases} 0 & if\ \beta \in \{0,1,2,\cdots,m-1\} \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)^{\beta-\alpha} & if\ \beta \in \mathbb{N}\ and\ \beta \geq m, \\ & or\ \beta \notin \mathbb{N}\ and\ \beta > m-1 \end{cases} $$

We adopt Caputo's definition, which is a modification of the R-L definition and has the advantage of dealing properly with initial value problem, for the concept of the fractional derivative.

Moreover, in this part we display two important formulas: Multinomial formula, is a natural extension of binomial formula (Berge, 1971; Merris, 2003) and generalized Taylor formula in the Caputo fractional derivative sense (Odibat *et al.*, 2007), respectively:

Multinomial Theorem (1):

Let $x_1, x_2, \cdots x_m \in \mathbb{R}$ and $n \in \{0,1,2,\cdots\}$ and let K be the set of non-negative integer solutions to the equation $k_1 + k_2 + \cdots + k_m = n$, that is $K = \{(k_1, k_2, \cdots, k_m)\}: k_i \in \mathbb{Z}$ and $k_i \geq 0$, $\forall\ i \in \{1,2,\cdots,m\}$ and $\sum_{i=1}^{m} k_i = n$. Then the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ is given by

$$ (x_1 + x_2 + \cdots + x_m)^n = \sum_{(k_1,k_2,\cdots,k_m)\in K} \binom{n}{k_1, k_2, \cdots, k_m} \prod_{i=1}^{m} {x_i}^{k_i} \quad \ldots(2) $$

Since k_i 's are nonnegative integers and the sum $(\sum\ \ )$ here is taken over all nonnegative $k_1, k_2, \cdots, k_m$ for which $k_1 + k_2 + \cdots + k_m = n$.
On the other hand, we can see that

$$ n = (n-k_1) + (k_1 - k_2) + (k_2 - k_3) + \cdots + (k_{m-2} - k_{m-1}) + (k_{m-1}) $$

As well as the multinomial coefficient $M[n; k_1, k_2, \cdots, k_m]$ is calculate by (3):

$$M[n;k_1,k_2,\cdots,k_m]=\binom{n}{k_1,k_2,\cdots,k_m}=\binom{n}{k_1}\binom{k_1}{k_2}\binom{k_2}{k_3}\cdots\binom{k_{m-3}}{k_{m-2}}\binom{k_{m-2}}{k_{m-1}}$$
$$=\frac{n!}{(n-k_1)!\,(k_1-k_2)!\,(k_2-k_3)!\cdots(k_{m-3}-k_{m-2})!\,(k_{m-2}-k_{m-1})!\,(k_{m-1})!}\quad\ldots(3)$$

After (3), we can rewrite formula (2) as follows:

$$(x_1+x_2+\cdots+x_m)^n=\left[\sum_{i=1}^{m}x_i\right]^n$$
$$=\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_2}\cdots\sum_{k_{m-2}=0}^{k_{m-3}}\sum_{k_{m-1}=0}^{k_{m-2}}M[n;k_1,k_2,\cdots,k_m]\,x_1^{n-k_1}x_2^{k_1-k_2}x_3^{k_2-k_3}\cdots x_{m-1}^{k_{m-2}-k_{m-1}}x_m^{k_{m-1}}\quad\ldots(4)$$

Generalized Taylor Theorem (2):

Suppose that ${}_a^CD_x^{k\alpha}f(x)\in C[a,b]$, for $k=0,1,2,3,\ldots,n+1$; where $0<\alpha\le 1$, then we have

$$f(x)=\sum_{i=0}^{n}\frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}{}_a^CD_x^{i\alpha}f(x=a)+\frac{{}_a^CD_x^{(n+1)\alpha}f(x=\xi)}{\Gamma\big((n+1)\alpha+1\big)}(x-a)^{(n+1)\alpha}\quad\ldots(5)$$

With $a\le\xi\le x$; $\forall\, x\in[a,b]$ and ${}_a^CD_x^{n\alpha}={}_a^CD_x^{\alpha}\,{}_a^CD_x^{\alpha}\ldots{}_a^CD_x^{\alpha}(n-\text{times})$.

3. FUNDAMENTAL MATRIX REPRESENTATIONS:

Let us first recall the multi-higher order nonlinear integro-fractional differential equations of Fredholm integer power Hammerstein type with unknown coefficients (1) and the fractional order of form $i\alpha; 0<\alpha\le 1$, and $\beta_j\in\mathbb{R}^+,\forall i;j\in\mathbb{Z}^+$:

$${}_a^CD_t^{n\alpha}y(t)+\sum_{i=1}^{n-1}\mathcal{P}_i(t)\,{}_a^CD_t^{(n-i)\alpha}y(t)+\mathcal{P}_n(t)y(t)=f(t)+\sum_{j=0}^{m}\lambda_j\int_a^b\mathcal{K}_j(t,s)\left[{}_a^CD_s^{\beta_j}y(s)\right]^{\ell_j}ds\quad\ldots(6)$$

with the mixed conditions of the form for all $k=0,1,\cdots,\mu-1$:

$$\sum_{j=0}^{\mu-1}\sum_{i=0}^{R}A_{kj}^{i}y^{(j)}(a_i)=c_k\quad\ldots(7)$$

where $R\in\mathbb{Z}^+$ and $a\le a_i\le b$ for all $i=0,1,\cdots,R$ with $\mu=\max\{\lceil n\alpha\rceil,\lceil\beta_j\rceil:j=0,1,\cdots,m\}$ and specially, if we put $R=2$ and $a_0=a$; $a_1=b$, $A_{kj}^0=e_{kj}$, $A_{kj}^1=d_{kj}$ for all k and j we can formed equation (7) as a linear combination in equation (8):

$$\sum_{j=0}^{\mu-1}\left[e_{kj}y^{(j)}(a)+d_{kj}y^{(j)}(b)\right]=c_k\quad\cdots(8)$$

For all $k=0,1,\cdots,\mu-1$.

Firstly, we rewrite the equation (6) in the form

$$\mathfrak{D}^{\alpha}(t)=f(t)+\lambda\mathcal{F}^{\beta}(t)\quad\cdots(9)$$

where the sequential fractional differential part is:

$$\mathfrak{D}^{\alpha}(t)=\sum_{i=0}^{n}\mathcal{P}_i(t)\,{}_a^CD_t^{(n-i)\alpha}y(t)\quad\ldots(10)$$

with $\mathcal{P}_0(t)=1$; and the Fredholm integer power Hammerstein integro-fractional part is:

$$\mathcal{F}^{\beta}(t)=\sum_{j=0}^{m}\lambda_j\int_a^b\mathcal{K}_j(t,s)\mathcal{H}_j(s;\beta,\ell)ds\quad\cdots(11)$$

where β and ℓ are the β_j and ℓ_j, respectively for all $j=0,1,\cdots,m$ with symbolic

$$\mathcal{H}_j(s;\beta,\ell)=\left[{}_a^CD_s^{\beta_j}y(s)\right]^{\ell_j}\quad\cdots(12)$$

We assume that the solution of the given problem in the form of the expression (6 and 7) is a truncated generalized Taylor's series of α-Caputo fractional order. In the following parts, we express the solution $y(t)$ and it's *k-th* sequential $\alpha-$Caputo fractional derivative ${}_{a}^{C}D_{t}^{k\alpha}y(t)$, $\mathfrak{D}^{\alpha}(t)$ and $\mathcal{F}^{\beta}(t)$ and the mixed condition to matrix form:

3.1 MATRIX REPRESENTATIONS FOR y(t) AND ${}_{a}^{C}D_{t}^{r\alpha}y(t)$:

We first consider the desired solution $y(t)$ of equation (6) defined by the $\alpha-$Caputo generalized Taylor series about $t=\tau=a$:

$$y(t)=\sum_{r=0}^{\infty}\frac{\psi_k}{\Gamma(k\alpha+1)}(t-\tau)^{k\alpha} \quad \ldots(13)$$

where $\psi_k=[{}_{a}^{C}D_{t}^{k\alpha}y(t)]_{t=\tau}$. Then we can put the N-truncated series equation (13) in the matrix form:

$$[y(t)]=T^{\alpha}\mathcal{M}_0\Psi \quad \ldots(14)$$

where

$$\left.\begin{array}{c} T^{\alpha}=[1 \quad (t-\tau)^{\alpha} \quad (t-\tau)^{2\alpha} \quad \cdots \quad (t-\tau)^{N\alpha}]_{1\times(N+1)} \\ \text{and} \\ \Psi=[\psi_0 \quad \psi_1 \quad \cdots \quad \psi_N]^{T}_{(N+1)\times 1} \end{array}\right\} \quad \ldots(15)$$

with $\mathcal{M}_0$ is define as follow:

$$\mathcal{M}_0=\begin{bmatrix} \frac{1}{\Gamma(1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\Gamma(1+\alpha)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\Gamma(1+2\alpha)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\Gamma(1+N\alpha)} \end{bmatrix}_{(N+1)\times(N+1)} \quad \ldots(16)$$

Let as assume that the $r\alpha$-*th* Caputo-fractional derivative of equation (13) with respect to t has the following generalized Taylor series expansion about $t=\tau=a$:

$${}_{a}^{C}D_{t}^{r\alpha}y(t)=\sum_{k=0}^{\infty}\frac{\psi_k^{(r)}}{\Gamma(k\alpha+1)}(t-\tau)^{k\alpha} \quad \ldots(17)$$

Where $r=0$, we have ${}_{a}^{C}D_{t}^{0}yt=y(t)$ and $\psi_k^{(0)}=\psi_k$ for all $k=0,1,2,\ldots$.

Apply sequential fractional derivative property of α-Caputo fractional definition on the equation (17), we obtain

$$\begin{aligned} {}_{a}^{C}D_{t}^{(r+1)\alpha}y(t) &= {}_{a}^{C}D_{t}^{\alpha}\left({}_{a}^{C}D_{t}^{r\alpha}y(t)\right) \\ &= \sum_{k=1}^{\infty}\frac{\psi_k^{(r)}}{\Gamma(k\alpha+1)}\,{}_{a}^{C}D_{t}^{\alpha}(t-\tau)^{k\alpha} \\ &= \sum_{k=1}^{\infty}\frac{\psi_k^{(r)}}{\Gamma\left((k-1)\alpha+1\right)}(t-\tau)^{(k-1)\alpha} \end{aligned}$$

Interchange (k by $k-1$), we obtain for all $r\geq 0$:

$${}_{a}^{C}D_{t}^{(r+1)\alpha}y(t)=\sum_{k=0}^{\infty}\frac{\psi_{k+1}^{(r)}}{\Gamma(k\alpha+1)}(t-\tau)^{k\alpha} \ldots(18)$$

On the other hand, rewrite equation (17) for $(r+1)$-th time α-sequential fractional derivative:

$${}_{a}^{C}D_{t}^{(r+1)\alpha}y(t)=\sum_{k=0}^{\infty}\frac{\psi_k^{(r+1)}}{\Gamma(k\alpha+1)}(t-\tau)^{k\alpha} \ldots(19)$$

From the relation (18) and (19), we obtain the recurrence relation between the generalized Taylor coefficients $\psi_k^{(r)}$ and $\psi_k^{(r+1)}$ of ${}_{a}^{C}D_{t}^{r\alpha}y(t)$ and ${}_{a}^{C}D_{t}^{(r+1)\alpha}y(t)$, respectively :

$$\psi_k^{(r+1)}=\psi_{k+1}^{(r)}\,; \quad r \text{ and } k=0,1,2,\ldots \ldots(20)$$

Now we take $k=0,1,\ldots,N$, (is a number of the truncated Taylor expansion) and assume $\psi_k^{(r)}=0$ for $k>N$. Then system (20) can be transformed in to the matrix form

$$\Psi^{(r+1)}=\mathcal{M}\Psi^{(r)}, \qquad r=0,1,2,\ldots \quad \ldots(21)$$

where

$$\Psi^{(r)}=\left[\Psi_0^{(r)} \quad \Psi_1^{(r)} \quad \ldots \quad \Psi_N^{(r)}\right]^{T} \quad \ldots(22a)$$

and

$$\mathcal{M}=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)} \quad \ldots(22b)$$

For $r=0,1,2,\cdots$ it follows form relation (21) that we have the following matrix relation:

$$\left.\begin{array}{c} \Psi^{(1)}=\mathcal{M}\Psi^{(0)}=\mathcal{M}\Psi \\ \Psi^{(2)}=\mathcal{M}\Psi^{(1)}=\mathcal{M}(\mathcal{M}\Psi)=\mathcal{M}^2\Psi \\ \vdots \\ \Psi^{(r)}=\mathcal{M}\Psi^{(r-1)}=\mathcal{M}^r\Psi \end{array}\right\} \ldots(23)$$

where clearly

$$\Psi^{(0)}=\Psi=[\psi_0 \quad \psi_1 \quad \cdots \quad \psi_N]^{T}$$

From the, matrix equation (23) obtain the relation between the N-truncated Taylor coefficient matrix Ψ of $y(t)$ and the N-truncated Taylor coefficient matrix $\Psi^{(r)}$ of the $r\alpha$-th Caputo derivative of $y(t)$ at equation (17); for all $r = 0,1,2,\cdots N$:

$$\begin{aligned} {}_a^C D_t^{r\alpha} y(t) &= T^{\alpha}(t)\mathcal{M}_0 \Psi^{(r)} \\ &= T^{\alpha}(t)[\mathcal{M}_0 \mathcal{M}^r]\Psi \\ &= T^{\alpha} \bar{\mathcal{M}}_r \Psi \end{aligned} \quad \cdots (24)$$

where

$$\bar{\mathcal{M}}_r = \mathcal{M}_0 \mathcal{M}^r \;;\; \mathcal{M}_0 = I \quad \ldots (25)$$

In addition, to compute the generalized Taylor coefficients we use the collocation points defined by

$$t_i = a + i\frac{b-a}{N} \;;\quad i = 0,1,\cdots,N \quad \cdots (26)$$

Where the values t_i are spread out over the interval $[a,b]$ and satisfy $a = t_0 < t_1 < \cdots <$

Where the values t_i are spread out over the interval $[a,b]$ and satisfy $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ substitute the Taylor collocation points $\{t_i\}_{i=0}^{N}$ into equation (24) for each $i = 0,1,\cdots,N$, we obtain:

$$[{}_a^C D_t^{r\alpha} y(t)]_{t=t_i} = T^{\alpha}(t_i)\bar{\mathcal{M}}_r \Psi;\; r = \overline{0{:}N} \cdots (27)$$

So that, the new matrix form is formed as below

$$\boldsymbol{Y}[r,\alpha] = \mathcal{C}^{\alpha} \bar{\mathcal{M}}_r \Psi \quad \ldots (28)$$

where

$$\boldsymbol{Y}[r,\alpha] = \left[[{}_a^C D_t^{r\alpha} y(t)]_{t=t_0} \quad [{}_a^C D_t^{r\alpha} y(t)]_{t=t_1} \quad \cdots \quad [{}_a^C D_t^{r\alpha} y(t)]_{t=t_N}\right]^T_{((N+1)\times 1)} \quad \ldots (29)$$

and

$$\mathcal{C}^{\alpha} = \begin{bmatrix} T^{\alpha}(t_0) \\ T^{\alpha}(t_1) \\ \vdots \\ T^{\alpha}(t_N) \end{bmatrix} = \begin{bmatrix} 1 & (t_0-\tau)^{\alpha} & (t_0-\tau)^{2\alpha} & \cdots & (t_0-\tau)^{N\alpha} \\ 1 & (t_1-\tau)^{\alpha} & (t_1-\tau)^{2\alpha} & & (t_1-\tau)^{N\alpha} \\ & \vdots & & \ddots & \vdots \\ 1 & (t_N-\tau)^{\alpha} & (t_N-\tau)^{2\alpha} & \cdots & (t_N-\tau)^{N\alpha} \end{bmatrix}_{(N+1)\times(N+1)} \quad \ldots (30)$$

Note, as a special type, if $r = 0$ in equation (27) and using the definition of Caputo fractional we obtain the equation (14) in collocation points terms.

3.2 MATRIX REPRESENTATION FOR FRACTIONAL DIFFERENTIAL PART $\mathfrak{D}^{\alpha}(t)$:

To make a matrix form of α-Caputo fractional differential $\mathfrak{D}^{\alpha}(t)$, define in relation (10), by putting the Taylor collocation points (26) in it we get the following equation, for all $k = 0,1,\ldots,N$:

$$\mathfrak{D}^{\alpha}(t_k) = \sum_{i=0}^{n} \boldsymbol{\mathcal{P}}_i(t_k)\left[{}_a^C D_t^{(n-i)\alpha} y(t)\right]_{t=t_k}$$

This system can be written in the matrix representation:

$$\mathfrak{D}^{\alpha} = \sum_{i=0}^{n} \boldsymbol{\mathcal{P}}_i \boldsymbol{Y}[(n-i),\alpha] \quad \ldots (31)$$

where

$$\left.\begin{aligned} \boldsymbol{\mathcal{P}}_i &= \mathrm{diag}[\mathcal{P}_i(t_0) \quad \mathcal{P}_i(t_1) \quad \cdots \quad \mathcal{P}_i(t_N)]_{(N+1)} \\ \mathfrak{D}^{\alpha} &= [\mathfrak{D}^{\alpha}(t_0) \quad \mathfrak{D}^{\alpha}(t_1) \quad \cdots \quad \mathfrak{D}^{\alpha}(t_N)]^T \end{aligned}\right\} \quad \ldots (32)$$

and $\mathcal{C}^{\alpha}$ with $\boldsymbol{Y}[n-i]$ are given in equations (30) with (29), respectively for all $r = \overline{0{:}n}$ and fractional orders α. Then relations (31) and (28) are formed the matrix relation of fractional part:

$$\mathfrak{D}^{\alpha} = \left\{\sum_{i=0}^{n} \boldsymbol{\mathcal{P}}_i \mathcal{C}^{\alpha} \bar{\mathcal{M}}_{n-i}\right\}\Psi \quad \ldots (33)$$

3.3 MATRIX REPRESENTATION FOR KERNEL TERMS:

In this part for each $j = 0,1,\ldots,m$, it is supposed that the kernel function $\lambda_j \mathcal{K}_j(t,s)$ in the Fredholm integral can be expanded to a N_*-truncated normal Taylor series about $s = t = \tau$, $(a \le \tau \le b)$, for $j = 0,1,\ldots,m$, in the form:

$$\lambda_j \mathcal{K}_j(t,s) = \sum_{r=0}^{N_*}\sum_{p=0}^{N_*} k_{rp}^{j}(t-\tau)^r(s-\tau)^p \quad \ldots (34)$$

Where, for each $r,p = 0,1,\ldots,N_*$:

$$k_{rp}^{j} = \frac{1}{r!\,p!}\frac{\partial^{r+p}\left[\lambda_j \mathcal{K}_j(t,s)\right]}{\partial t^r \partial s^p}\bigg|_{(t=\tau,s=\tau)} \quad \ldots (35)$$

Expression (34), for each $j = 0,1,\ldots,m$, can be written in the following matrix form

$$\left[\lambda_j \mathcal{K}_j(t,s)\right] = T(t)K_j[(S(s)]^T \quad \ldots (36)$$

where

$$T(t) = [1 \quad (t-\tau) \quad (t-\tau)^2 \quad \cdots \quad (t-\tau)^{N_*}]_{(N_*+1)\times 1}$$

$$S(s) = [1 \quad (s-\tau) \quad (s-\tau)^2 \quad \cdots \quad (s-\tau)^{N_*}]_{(N_*+1)\times 1}$$

and

$$K_j = [k_{rp}^j] = \begin{bmatrix} k_{00}^j & k_{01}^j & \cdots & k_{0N_*}^j \\ k_{10}^j & k_{11}^j & & k_{1N_*}^j \\ \vdots & & \ddots & \vdots \\ k_{N_*0}^j & k_{N_*1}^j & \cdots & k_{N_*N_*}^j \end{bmatrix}_{(N_*+1)\times(N_*+1)} \quad \ldots(37)$$

3.4 MATRIX REPRESENTATION FOR HAMMERSTEIN TERMS $\mathcal{H}_j(s;\beta,\ell)$:

In this part we use Cauchy product of ℓ_j , $j = 0,1,\ldots,m$, series (Multinomial theorem 1). First, recall the equation (12) and approximate the unknown function $y(t)$ by N-truncated generalized Taylor series around $t = \tau = a$ then apply the linearty property of Caputo-fractional operator with equation to get the following formula for each $j = 0,1,\cdots,m$:

$$\mathcal{H}_j(s;\beta,\ell) = \left[{}_a^C D_s^{\beta_j} y(s)\right]^{\ell_j} = \left[{}_a^C D_s^{\beta_j} \sum_{k=0}^{N} \psi_k (s-\tau)^{k\alpha}\right]^{\ell_j}$$

So:

$$\mathcal{H}_j(s;\beta,\ell) = \left[\sum_{k=1}^{N} \bar{\psi}_k (s-\tau)^{k\alpha-\beta_j}\right]^{\ell_j} \quad \ldots(38)$$

where, for all $k = 1,2,\ldots,N$ we define $\bar{\psi}_k$ by:

$$\bar{\psi}_k = \frac{\psi_k}{\Gamma(k\alpha - \beta_j + 1)} \quad \ldots(39)$$

Now apply Multinomial Theorem (1) on equation (38) to obtain for each $j = 0,1,\ldots,m$:

$$\mathcal{H}_j(s;\beta,\ell) = \sum_{k_1=0}^{\ell_j} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{N-2}=0}^{k_{N-3}} \sum_{k_{N-1}=0}^{k_{N-2}} M[\ell_j; k_1, k_2, \cdots, k_N] * \\ * \left[\bar{\psi}_1 (s-\tau)^{\alpha-\beta_j}\right]^{\ell_j - k_1} \left[\bar{\psi}_2 (s-\tau)^{2\alpha-\beta_j}\right]^{k_1-k_2} \left[\bar{\psi}_3 (s-\tau)^{3\alpha-\beta_j}\right]^{k_2-k_3} * \cdots \\ * \left[\bar{\psi}_{N-1} (s-\tau)^{(N-1)\alpha-\beta_j}\right]^{k_{N-2}-k_{N-1}} \left[\bar{\psi}_N (s-\tau)^{N\alpha-\beta_j}\right]^{k_{N-1}}$$

where (3) is define M for each j as:

$$M[\ell_j; k_1, k_2, \cdots, k_N] = \binom{\ell_j}{k_1}\binom{k_1}{k_2}\binom{k_2}{k_3}\cdots\binom{k_{N-3}}{k_{N-2}}\binom{k_{N-2}}{k_{N-1}} \\ = \frac{\ell_j}{(\ell_j - k_1)!\,(k_1-k_2)!\,(k_2-k_3)!\cdots(k_{N-3}-k_{N-2})!\,(k_{N-2}-k_{N-1})!\,(k_{N-1})!} \quad \cdots(40)$$

So that we express it in the following matrix form:

$$\mathcal{H}_j(s;\beta,\ell) = \bar{S}_{\ell_j}^{\alpha,\beta_j}(s)\bar{\Psi}_{\ell_j}^{\alpha,\beta_j} \quad \ldots(41)$$

where

$$\bar{\Psi}_{\ell_j}^{\alpha,\beta_j} = \begin{bmatrix} \sum_{k_1=0}^{0}\sum_{k_2=0}^{k1}\sum_{k_3=0}^{k_2}\cdots\sum_{k_{\ell_j-1}=0}^{k_{\ell_j}-2} \bar{\psi}_{k_{\ell_j-1}+1}\bar{\psi}_{k_{\ell_j-2}-k_{\ell_j-1}+1}\cdots\bar{\psi}_{k1-k_2+1}\bar{\psi}_{1-k1} \\ \sum_{k_1=0}^{1}\sum_{k_2=0}^{k1}\sum_{k_3=0}^{k_2}\cdots\sum_{k_{\ell_j-1}=0}^{k_{\ell_j}-2} \bar{\psi}_{k_{\ell_j-1}+1}\bar{\psi}_{k_{\ell_j-2}-k_{\ell_j-1}+1}\cdots\bar{\psi}_{k1-k_2+1}\bar{\psi}_{2-k1} \\ \vdots \\ \sum_{k_1=0}^{\ell_j(N-1)}\sum_{k_2=0}^{k1}\sum_{k_3=0}^{k_2}\cdots\sum_{k_{\ell_j-1}=0}^{k_{\ell_j}-2} \bar{\psi}_{k_{\ell_j-1}+1}\bar{\psi}_{k_{\ell_j-2}-k_{\ell_j-1}+1}\cdots\bar{\psi}_{k1-k_2+1}\bar{\psi}_{\ell_j(N-1)+1-k1} \end{bmatrix}_{(\ell_j(N-1)+1)\times 1} \quad \ldots (42)$$

That $\bar{\psi}_p = 0$ for all $p > N$ and

$$\bar{S}_{\ell_j}^{\alpha,\beta_j}(s) = (s-\tau)^{\ell_j(\alpha-\beta_j)}[1 \quad (s-\tau)^{\alpha} \quad \cdots \quad (s-\tau)^{\ell_j(N-1)\alpha}]_{1\times(\ell_j(N-1)+1)} \quad \ldots (43)$$

3.5 MATRIX REPRESENTATION FOR THE FREDHOLM INTEGRO-FRACTIONAL DIFFERENTIAL PART $\boldsymbol{\mathcal{F}^{\beta}(t)}$:

Substituting the matrix forms (36) and (41) corresponding to the functions $\mathcal{K}_j(t,s)$ and $\mathcal{H}_j(s;\beta,\ell)$, respectively, into the equation (11) and then simplifying the result equation, to obtain the matrix representation of $\mathcal{F}^{\beta}(t)$:

$$[\mathcal{F}^{\beta}(t)] = \int_a^b \left\{T(t)\sum_{j=0}^{m} K_j[S(s)]^T \bar{S}_{\ell_j}^{\alpha,\beta_j}(s)\bar{\Psi}_{\ell_j}^{\alpha,\beta_j}\right\} ds$$

$$= T(t)\sum_{j=0}^{m} K_j\left\{\int_a^b [S(s)]^T \bar{S}_{\ell_j}^{\alpha,\beta_j}(s)\, ds\right\}\bar{\Psi}_{\ell_j}^{\alpha,\beta_j}$$

$$= T(t)\left(\sum_{j=0}^{m} K_j \mathrm{Q}_{\ell_j}\bar{\Psi}_{\ell_j}^{\alpha,\beta_j}\right) \quad \ldots (44)$$

where

$$\mathrm{Q}_{\ell_j} = \int_a^b [S(s)]^T \bar{S}_{\ell_j}^{\alpha,\beta_j}(s)\, ds = [q_{ik}^{\ell_j}]$$

while

$$q_{ik}^{\ell_j} = \left.\frac{(b-\tau)^{(\ell_j+k)\alpha-\ell_j\beta_j+i+1} - (a-\tau)^{(\ell_j+k)\alpha-\ell_j\beta_j+i+1}}{(\ell_j+k)\alpha-\ell_j\beta_j+i+1}\right|_{\substack{k=0,1,\ldots,\ell_j(N-1) \\ i=0,1,\ldots,N_* \\ j=1,2,\ldots,m}} \quad \ldots (45)$$

Then substituting the Taylor collocation points (26) in to equation (44), we get the matrix forms:

$$\boldsymbol{\mathcal{F}}^{\beta} = \boldsymbol{C}\left(\sum_{j=0}^{m} \boldsymbol{K}_j \boldsymbol{Q}_{\ell_j}\bar{\boldsymbol{\Psi}}_{\ell_j}^{\alpha,\beta_j}\right) \quad \ldots (46)$$

where $\boldsymbol{C}$, $\mathbf{K}_j$, $\boldsymbol{Q}_{\ell_j}$ *and* $\bar{\boldsymbol{\Psi}}_{\ell_j}^{\alpha,\beta_j}$ matrices and all can be defined as in equations (46a):

$$\left.\begin{aligned} \boldsymbol{K}_j &= \mathrm{diag}[K_j \quad K_j \quad \cdots \quad K_j]_{(m+1)(N+1)} \\ \boldsymbol{C} &= \mathrm{diag}[T_0 \quad T_1 \quad \cdots \quad T_N]_{(N+1)} \\ \boldsymbol{Q}_{\ell_j} &= \mathrm{diag}[\mathrm{Q}_{\ell_j} \quad \mathrm{Q}_{\ell_j} \quad \cdots \quad \mathrm{Q}_{\ell_j}]_{(N+1)} \\ \bar{\boldsymbol{\Psi}}_{\ell_j}^{\alpha,\beta_j} &= \mathrm{diag}[\bar{\Psi}_{\ell_j}^{\alpha,\beta_j} \quad \bar{\Psi}_{\ell_j}^{\alpha,\beta_j} \quad \cdots \quad \bar{\Psi}_{\ell_j}^{\alpha,\beta_j}]_{(N+1)} \end{aligned}\right\}$$

Simplify

$\ldots (46a)$

$$T_i = [1 \quad (t_i-\tau) \quad (t_i-\tau)^2 \quad \cdots \quad (t_i-\tau)^{N_*}]_{1\times(N_*+1)}$$

3.6 MATRIX REPRESENTATION FOR THE CONDITIONS:

Now we obtain to find the matrix representation of the mixed conditions (7) and using the definition of Caputo-derivative, we can write

$$D^j y(t) = {}_a^C D_t^{j\left(\frac{1}{\alpha}\right)\alpha} y(t)$$

for all j.

Then, we write $\ell_j = \frac{j}{\alpha}$ and formed $\alpha = \frac{p}{q}$ $(p, q \in \mathbb{R}^+)$ provided that p divided s jq for all j. We can express the expression (14) and its derivative (24) as:

$$D^j y(t)\Big|_{t=a_i} = {}_a^C D_t^{j\left(\frac{1}{\alpha}\right)\alpha} y(t)\Bigg|_{t=a_i}$$

$$= {}_a^C D_t^{\ell_j \alpha} y(t)\Bigg|_{t=a_i} = T^\alpha(a_i)\bar{\mathcal{M}}_{\ell_j}\Psi \quad \cdots (47)$$

where

$$T^\alpha(a_i) = [1 \quad (a_i - \tau)^\alpha \quad (a_i - \tau)^{2\alpha} \quad \cdots \quad (a_i - \tau)^{N\alpha}]$$

and, for all j

$$\bar{\mathcal{M}}_{\ell_j} = \mathcal{M}_0 \mathcal{M}^{\ell_j}, \quad \mathcal{M}_0 = \mathrm{I}$$

By replacing (47) in to condition (7) and then simplify, for all $k = 0,1,\cdots,\mu - 1$ we have

$$U_k\Psi = [C_k] \quad \ldots (48)$$

where

$$U_k = \sum_{j=0}^{\mu-1}\left[\sum_{i=0}^{R} A_{kj}^i T^\alpha(a_i)\right]\bar{\mathcal{M}}_{\ell_j} = [u_{k0} \quad u_{k1} \quad \cdots \quad u_{kN}]$$

From here, if $\mu = 1 = R$ with take $a_0 = a$; $a_1 = b$ and $\bar{\mathcal{M}}_{\ell_0} = \mathcal{M}_0$ $(\ell_0 = 0)$ so we have one equation from (48):

$$U_0\Psi = [C_0]$$

$$U_0 = [A_{00}^0 T^\alpha(a) + A_{00}^1 T^\alpha(b)]\mathcal{M}_0 \quad \cdots (49)$$

If $\mu = 2$; $R = 1$ with take $a_0 = a$; $a_1 = b$ and $\bar{\mathcal{M}}_{\ell_0} = \mathcal{M}_0$; $\bar{\mathcal{M}}_{\ell_1} = \mathcal{M}_0\mathcal{M}^{\ell_0}$; also let

$$A_{00}^0 + A_{01}^0 = e_{00}, A_{00}^1 + A_{01}^1 = d_{00}$$
$$A_{10}^0 + A_{11}^0 = e_{11}, A_{10}^1 + A_{11}^1 = d_{11}$$

so we have two equations from (48):

$$U_k\Psi = [C_k]\,; \quad k = 0 \text{ and } 1 \quad \cdots (50)$$

In more details

$$U_0 = [e_{00}T^\alpha(a) + d_{00}T^\alpha(b)]\mathcal{M}_0$$

$$U_1 = [e_{11}T^\alpha(a) + d_{11}T^\alpha(b)]\mathcal{M}_0\mathcal{M}^{\ell_1}$$

4. THE PROCESS OF THE GENERALIZED TAYLOR'S COLLOCATION METHOD TO SOLVE NON-LINEAR IFDES OF F-H TYPE BY USING MATRIX REPRESENTATION:

In this section, we are going to construct the fundamental matrix equation corresponding to multi high-order nonlinear IFDEs of Fredholm-Hammerstein type (6) and mixed boundary conditions (7). For this purpose, substituting the matrix relations (33), (36) and (46) into equation (9) and after some simple manipulations we gain the fundamental matrix equation as:

$$\left\{\sum_{i=0}^{n} \boldsymbol{\mathcal{P}}_i \mathcal{C}^\alpha \bar{\mathcal{M}}_{n-i}\right\}\boldsymbol{\Psi} - \left\{\boldsymbol{\mathcal{C}}\sum_{j=0}^{m} \boldsymbol{K}_j \boldsymbol{Q}_{\ell_j} \bar{\boldsymbol{\Psi}}_{\ell_j}^{\alpha,\beta_j}\right\} = \boldsymbol{F}$$

where ... (51)

$$\boldsymbol{F} = [f(t_0) \quad f(t_1) \quad \cdots \quad f(t_N)]^T \quad \ldots (52)$$

and $\boldsymbol{\mathcal{P}}_i, \mathcal{C}^\alpha, \bar{\mathcal{M}}_{n-i}, \boldsymbol{C}, \boldsymbol{K}_j, \boldsymbol{Q}_{\ell_j}$ and $\boldsymbol{\Psi} = \Psi$ with $\bar{\boldsymbol{\Psi}}_{\ell_j}^{\alpha,\beta_j}$ are defined in equations (32, 30, 25, 46a, 45, 15, 46a) respectively.

The relation matrix equation (51) corresponds to a system of $(N + 1)$ non-linear algebraic equations for the $(N + 1)$-unknown coefficients $\left(\Psi_k = {}_a^C D_t^{k\alpha} y(t) \text{ at } t = \tau = a\right), k = 0,1,\ldots,N$ and we can write it, for all $j = 0,1,\cdots,m$, in the form:

$$W\boldsymbol{\Psi} - V\bar{\boldsymbol{\Psi}}_{\ell_j}^{\alpha,\beta_j} = \boldsymbol{F} \quad \ldots (53)$$

or, $[W; V: \boldsymbol{F}]$, where

$$W = [w_{hr}] = \sum_{i=0}^{n} \boldsymbol{\mathcal{P}}_i \mathcal{C}^\alpha \bar{\mathcal{M}}_{n-i}\,; h, r = 0,1,\ldots,N$$

$$V = [v_{hs}] = \boldsymbol{\mathcal{C}}\sum_{j=0}^{m} \boldsymbol{K}_j \boldsymbol{Q}_{\ell_j}$$

$$h = 0,1,\ldots,N; s = 0,1,\ldots,(N-1)\ell_j; \left(\text{for each } \ell_j\right)$$

and $\boldsymbol{F}$ is defined in equation (52). We can write the following form that express the matrix equation (46) which corresponds to the mixed conditions (7); for $k = 0,1,\cdots,\mu - 1$

$$[U_k\,;\, C_k] = [u_{k0} \quad u_{k1} \quad \cdots \quad u_{kN} \quad ; \quad c_k] \quad \cdots (54)$$

Consequently, to obtain the unknown terms of generalized truncated Taylor's polynomial (13), ${}_a^C D_t^{k\alpha} y(t)\big|_{t=\tau}$ for all $k = 0,1,\cdots,N$, which is the approximate solution of equation (6) under the mixed conditions (7). we now have the new block matrix by replacing the μ-row matrices (53) by the μ-rows of the matrix (54), we have the required augmented matrix:

$$[\widetilde{W};\widetilde{V}:\widetilde{F}] = \left[\begin{array}{cccc|cccc|c|cccc|c} w_{0,0} & w_{0,1} & \cdots & w_{0,N} & v_{0,0} & v_{0,1} & \cdots & v_{0,(N-1)\ell_0} & \cdots & v_{0,0} & v_{0,1} & \cdots & v_{0,(N-1)\ell_m} & f_0 \\ w_{1,0} & w_{1,1} & \cdots & w_{1,N} & v_{1,0} & v_{1,1} & \cdots & v_{1,(N-1)\ell_0} & \cdots & v_{1,0} & v_{1,1} & \cdots & v_{1,(N-1)\ell_m} & f_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N-\mu,0} & w_{N-\mu,1} & \cdots & w_{N-\mu,N} & v_{N-\mu,0} & v_{N-\mu,1} & \cdots & v_{N-\mu,(N-1)\ell_0} & \cdots & v_{N-\mu,0} & v_{N-\mu,1} & \cdots & v_{N-\mu,(N-1)\ell_m} & f_{N-\mu} \\ u_{0,0} & u_{0,1} & \cdots & u_{0,N} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & c_0 \\ u_{1,0} & u_{1,1} & \cdots & u_{1,N} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & c_1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ u_{\mu-1,0} & u_{\mu-1,1} & \cdots & u_{\mu-1,N} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & c_{\mu-1} \end{array}\right] \quad \ldots (55)$$

By using this nonlinear system, and applying Newton-Raphson iterative method, (Richard *et al.*, 2011 and Walter, 2012); the unknown N-truncated generalized Taylor coefficients $\psi_k = {}_a^C D_t^{k\alpha} y(t) \ at \ \ t = \tau, \ \ k = 0,1,\cdots,N$, are obtained. When it is placed into the relation (13) the approximation solution $y(t)$ is found of IFDEs of F-H types (6) with mixed conditions (7) which is unique.

The Algorithm [AGTM-NLFH**]:**

The approximate solution of IFDEs of F-H type (2) using generalized Taylor method can by summarized in the following stage:

Step 1:

a. Input the number of truncated generalized Taylor series N such that($N \geq \max\{n,m\}$) and truncated normal Taylor series N_*.

b. Assume $h = (b-a)/N$, $(N \in \mathbb{N})$.

c. Put C_i boundary conditions $i = 0,1,\ldots,\mu-1$; $\mu = \max\{\lceil n\alpha \rceil, \lceil \beta_j \rceil\}$ for all $j = 0,1.\cdots,m$.

Step 2: Determine the matrices $\overline{\mathcal{M}}_k = \mathcal{M}_0\mathcal{M}^k$ for $k = 0,1,\cdots,n$ and K_j for $j = 0,1,\cdots,m$ from the equations (16) , (22a) and (37) respectively.

Step 3:

a. Set the collocation points $t_k = a + \frac{b-a}{N}k$, $k = \overline{0:N}$

b. Evaluate the matrices $\boldsymbol{\mathcal{P}}_i(i = 0,1,\ldots,n), \boldsymbol{F}$ and $\mathcal{C}^\alpha$ from the equations (32), (52) and (30) respectively.

c. Compute the matrix $\boldsymbol{Q}_{\ell_j} = \left[q_{ik}^{\ell j}(t)\right]$ from equation (45) for all $\left(k = \overline{0:\ell_j(N-1)}\right)$ and each $i = \overline{0:N_*}$; $j = \overline{0:m}$.

Step 4: Construct the conditional μ-row matrix $U_k(k = 0,1,\ldots,\mu-1)$ from equation (54).

Step 5: Construct the matrices $\widetilde{W}$,$\widetilde{V}$and $\widetilde{\mathcal{F}}$ which are represented in equation (55).

Step 6: Solve the nonlinear system (55) for fractional generalized Taylor coefficients ψ_k $(k = 0,1,\ldots,N)$ using Newton's-Raphson-iterative method.

Step 7: Substitute all ψ_k 's into truncated generalized Taylor series (13 or 14) to obtain the approximate solution $\tilde{y}(t)$ of y(t).

5. NUMERICAL EXPERIMENTATIONS:

The method of this chapter is useful in finding the solution of non-linear IFDEs of F-H type in the terms of fractional generalized Taylor polynomials. In this section, several numerical examples are carried out in which the exact solution has already existed to show the effectiveness of the proposed algorithm [AGTM-NLFH]. All of them were performed on the computer using a program written in MatLab (V 9.2). the least square error in tables are the values of $\sum_{k=0}^{M}[y(t_k)-\tilde{y}_N(t_k)]^2$, selected points t_k and in Newton-iterative method choose the tolerance and number of iteration as an input data for all examples.

Example (1):

Let us consider the a high-order nonlinear IFDE of F-H Type with the unknown coefficients:

$$ {}_0^CD_t^{0.6}y(t)+2\,{}_0^CD_t^{0.2}y(t)+\sin(t)\,y(t) = f(t)+\int_0^1\{(s^2-2t)[{}_0^CD_s^{0.2}y(s)]^2 + (st+1)[{}_0^CD_s^{0.1}y(s)]^3\}ds $$

where

$$ f(t)=\frac{2}{\Gamma(1.4)}t^{0.4}+\frac{4}{\Gamma(1.8)}t^{0.8} + (1+2t)\sin(t) + \frac{4}{[\Gamma(1.8)]^2}\left(\frac{10}{13}t-\frac{5}{23}\right) - \frac{8}{[\Gamma(1.9)]^3}\left(\frac{10}{47}t+\frac{10}{37}\right) $$

with the boundary condition of form:

$$ y(0)-y(1)=-2 $$

which has the exact solution of expression $y(t)=1+2t$.

An approximate the solution $y(t)$ by truncated generalized Taylor series around $t=\tau=0$ and $(N=5)$:

$$ y(t)\cong\tilde{y}(t)=\sum_{k=0}^{5}\frac{\psi_k}{\Gamma(k\alpha+1)}(t-\tau)^{k\alpha}\ ; $$

$$ \psi_k=[{}_a^CD_t^{k\alpha}y(t)]_{t=0} $$

where

$$ \alpha=0.2\ ,\ \lambda_{1,2}=1, n=3\ , m=1\ , \quad a=0\ , b=1 $$

and

$$ \ell_1=2\ ,\ell_2=3, \qquad \beta_1=0.2\ ,\beta_2=0.1 $$

$$ \mathcal{P}_0(t)=1\ ,\quad \mathcal{P}_1(t)=0\ ,\quad \mathcal{P}_2(t)=2\ ,\quad \mathcal{P}_3(t)=\sin(t) $$

$$ \mathcal{K}_1(t,s)=s^2-2t\ ,\quad \mathcal{K}_2(t,s)=st+1 $$

Now, to solve the problem considering the collocation points for $N=5$ and $N_*=5$, using the algorithm AGTM-NLFH with running the MatLab program (TaylorMain) with all subprograms, the required matrices are:

$$ \mathcal{P}_0=I_6,\qquad \mathcal{P}_1=0_6, $$

$$ \mathcal{P}_2=\begin{bmatrix}2&0&0&0&0&0\\0&2&0&0&0&0\\0&0&2&0&0&0\\0&0&0&2&0&0\\0&0&0&0&2&0\\0&0&0&0&0&2\end{bmatrix}, $$

$$ \mathcal{P}_3=\begin{bmatrix}0&0&0&0&0&0\\0&0.198669&0&0&0&0\\0&0&0.389418&0&0&0\\0&0&0&0.564642&0&0\\0&0&0&0&0.717356&0\\0&0&0&0&0&0.841470\end{bmatrix} $$

$$ \bar{\mathcal{M}}_0=\mathcal{M}_0=\begin{bmatrix}\frac{1}{\Gamma(1)}&0&0&0&0&0\\0&\frac{1}{\Gamma(\alpha+1)}&0&0&0&0\\0&0&\frac{1}{\Gamma(2\alpha+1)}&0&0&0\\0&0&0&\frac{1}{\Gamma(3\alpha+1)}&0&0\\0&0&0&0&\frac{1}{\Gamma(4\alpha+1)}&0\\0&0&0&0&0&\frac{1}{\Gamma(5\alpha+1)}\end{bmatrix} $$

$$ \mathcal{M}=\begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix} $$

$$ \bar{\mathcal{M}}_1=\mathcal{M}_0\mathcal{M}^1=\begin{bmatrix}0&1&0&0&0&0\\0&0&1.089124&0&0&0\\0&0&0&1.127060&0&0\\0&0&0&0&1.119174&0\\0&0&0&0&0&1.073671\\0&0&0&0&0&0\end{bmatrix} $$

$$ \bar{\mathcal{M}}_2=\mathcal{M}_0\mathcal{M}^2=\begin{bmatrix}0&0&1&0&0&0\\0&0&0&1.089124&0&0\\0&0&0&0&1.127060&0\\0&0&0&0&0&1.119174\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix} $$

$$ \bar{\mathcal{M}}_3=\mathcal{M}_0\mathcal{M}^3=\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1.0891244&0\\0&0&0&0&0&1.1270605\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix} $$

and

$$\mathcal{C}^{\alpha}=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 1 & (0.2)^{\alpha} & (0.2)^{2\alpha} & (0.2)^{3\alpha} & (0.2)^{4\alpha} & (0.2)^{5\alpha}\\ 1 & (0.4)^{\alpha} & (0.4)^{2\alpha} & (0.4)^{3\alpha} & (0.4)^{4\alpha} & (0.4)^{5\alpha}\\ 1 & (0.6)^{\alpha} & (0.6)^{2\alpha} & (0.6)^{3\alpha} & (0.6)^{4\alpha} & (0.6)^{5\alpha}\\ 1 & (0.8)^{\alpha} & (0.8)^{2\alpha} & (0.8)^{3\alpha} & (0.8)^{4\alpha} & (0.8)^{5\alpha}\\ 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$$

Now, the matrix equation (33) of **the linear part** becomes

$$\{\mathcal{P}_0\mathcal{C}^{\alpha}\overline{\mathcal{M}}_3+\mathcal{P}_1\mathcal{C}^{\alpha}\overline{\mathcal{M}}_2+\mathcal{P}_2\mathcal{C}^{\alpha}\overline{\mathcal{M}}_1+\mathcal{P}_3\mathcal{C}^{\alpha}\overline{\mathcal{M}}_0\}\boldsymbol{\Psi}$$
$$=\begin{bmatrix}0 & 2 & 0 & 1 & 0 & 0\\ 0.198669 & 2.156824 & 1.696372 & 2.268756 & 1.700444 & 1.224335\\ 0.389418 & 2.353106 & 2.117727 & 2.813939 & 2.399340 & 1.968673\\ 0.564642 & 2.555240 & 2.485475 & 3.302660 & 3.033694 & 2.684552\\ 0.717356 & 2.747188 & 2.822639 & 3.763883 & 3.643733 & 3.400982\\ 0.841470 & 2.916466 & 3.126637 & 4.195874 & 4.230937 & 4.115874\end{bmatrix}\begin{bmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\\ \psi_4\\ \psi_5\end{bmatrix}$$

and $\mathcal{F}$ is a vector of order (1×6) defined by, from equation (52):

$$\mathcal{F}=[-3.432822\quad -0.458746\quad 1.547415\quad 3.481132\quad 5.393443\quad 7.274076]^{T}$$

Also, $\boldsymbol{\mathcal{C}},\boldsymbol{K}_j$ *and* $\boldsymbol{Q}_{\ell_j}$, for all $j=0,1$, are block matrices, from equations (46a) and (45) we can see the following matrices :

$$\boldsymbol{K}_j=\begin{bmatrix}K_j & 0\\ 0 & K_j\end{bmatrix},\qquad j=0,1$$

$$K_0=\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0\\ -2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$K_1=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$T(t_0=0.0)=[1\quad 0\quad 0\quad 0\quad 0\quad 0]^{T}$

$T(t_1=0.2)=[1\quad 0.2\quad 0.04\quad 0.008\quad 0.0016\quad 0.00032]^{T}$

$T(t_2=0.4)=[1\quad 0.4\quad 0.16\quad 0.064\quad 0.0256\quad 0.01024]^{T}$

$T(t_3=0.6)=[1\quad 0.6\quad 0.36\quad 0.216\quad 0.1296\quad 0.07776]^{T}$

$T(t_4=0.8)=[1\quad 0.8\quad 0.64\quad 0.512\quad 0.4096\quad 0.32768]^{T}$

$T(t_5=1.0)=[1\quad 1\quad 1\quad 1\quad 1\quad 1]^{T}$

and since $\ell_0=2$ and $\ell_1=3$ so from equation (45), we get $\boldsymbol{Q_2}$ and $\boldsymbol{Q_3}$:

$$\boldsymbol{\mathcal{C}}=\mathrm{diag}[T(t_0=0)\quad T(t_1=0.2)\quad T(t_2=0.4)\quad T(t_3=0.6)\quad T(t_4=0.8)\quad T(t_5=1)]$$

$$\boldsymbol{Q_2}=\begin{bmatrix}1.000000 & 0.833333 & 0.714285 & 0.625000 & 0.555556 & 0.500000 & 0.454545 & 0.416667 & 0.384615\\ 0.500000 & 0.454545 & 0.416667 & 0.384615 & 0.357143 & 0.333333 & 0.312500 & 0.294118 & 0.277778\\ 0.333333 & 0.312500 & 0.294118 & 0.277778 & 0.263158 & 0.250000 & 0.238095 & 0.227273 & 0.217391\\ 0.250000 & 0.238095 & 0.227273 & 0.217391 & 0.208333 & 0.200000 & 0.192308 & 0.185185 & 0.178571\\ 0.200000 & 0.192308 & 0.185185 & 0.178571 & 0.172414 & 0.166667 & 0.161290 & 0.15625 & 0.151515\\ 0.166667 & 0.161290 & 0.156250 & 0.151515 & 0.147059 & 0.142857 & 0.138889 & 0.135135 & 0.131579\end{bmatrix}$$

$$\boldsymbol{Q_3}=\begin{bmatrix}0.76923 & 0.66667 & 0.58824 & 0.52632 & 0.47619 & 0.43478 & 0.40000 & 0.37037 & 0.34483 & 0.32258 & 0.30303 & 0.28571 & 0.27027\\ 0.43478 & 0.40000 & 0.37037 & 0.34483 & 0.32258 & 0.30303 & 0.28571 & 0.27027 & 0.25641 & 0.24390 & 0.23256 & 0.22222 & 0.21277\\ 0.30303 & 0.28571 & 0.27027 & 0.25641 & 0.24390 & 0.23256 & 0.22222 & 0.21277 & 0.20408 & 0.19608 & 0.18868 & 0.18182 & 0.17544\\ 0.23256 & 0.22222 & 0.21277 & 0.20408 & 0.19608 & 0.18868 & 0.18182 & 0.17544 & 0.16949 & 0.16393 & 0.15873 & 0.15385 & 0.14925\\ 0.18868 & 0.18182 & 0.17544 & 0.16949 & 0.16393 & 0.15873 & 0.15384 & 0.14925 & 0.14493 & 0.14085 & 0.13699 & 0.13333 & 0.12987\\ 0.15873 & 0.15385 & 0.14925 & 0.14493 & 0.14085 & 0.13699 & 0.13333 & 0.12987 & 0.12658 & 0.12346 & 0.12048 & 0.11765 & 0.11494\end{bmatrix}$$

$\overline{\boldsymbol{\Psi}}_2^{0.2,0.2}$ and $\overline{\boldsymbol{\Psi}}_3^{0.2,0.1}$, are defined in the equation (42) and it is the vector respectively of order (9×1) and (13×1) formed as :

$$\overline{\boldsymbol{\Psi}}_2^{0.2,0.2}=\begin{bmatrix}\overline{\psi}_1^2\\ 2\overline{\psi}_1\overline{\psi}_2\\ \overline{\psi}_2^2+2\overline{\psi}_1\overline{\psi}_3\\ 2\overline{\psi}_1\overline{\psi}_4+2\overline{\psi}_2\overline{\psi}_3\\ \overline{\psi}_3^2+2\overline{\psi}_1\overline{\psi}_5+2\overline{\psi}_2\overline{\psi}_4\\ 2\overline{\psi}_2\overline{\psi}_5+2\overline{\psi}_3\overline{\psi}_4\\ \overline{\psi}_4^2+2\overline{\psi}_3\overline{\psi}_5\\ 2\overline{\psi}_4\overline{\psi}_5\\ \overline{\psi}_5^2\end{bmatrix}$$

where

$$\bar{\psi}_k = \frac{\psi_k}{\Gamma(k(0.2) - 0.2 + 1)} \qquad k = 1,2,\dots,5$$

And also, here we define $\bar{\psi}_k$ for $\bar{\boldsymbol{\Psi}}_3^{0.2,0.1}$ as:

$$\bar{\psi}_k = \frac{\psi_k}{\Gamma(k(0.2) - 0.1 + 1)} \qquad k = 1,2,\dots,5$$

$$\bar{\boldsymbol{\Psi}}_3^{0.2,0.1} = \begin{bmatrix} \bar{\psi}_1^3 \\ 3\bar{\psi}_1^2\bar{\psi}_2 \\ 3\bar{\psi}_1\bar{\psi}_2^2 + 3\bar{\psi}_1^2\bar{\psi}_3 \\ \bar{\psi}_2^3 + 3\bar{\psi}_1^2\bar{\psi}_4 + 6\bar{\psi}_1\bar{\psi}_2\bar{\psi}_3 \\ 3\bar{\psi}_1^2\bar{\psi}_5 + 6\bar{\psi}_1\bar{\psi}_2\bar{\psi}_4 + 3\bar{\psi}_1\bar{\psi}_3^2 + 3\bar{\psi}_2^2\bar{\psi}_3 \\ 3\bar{\psi}_2\bar{\psi}_3^2 + 3\bar{\psi}_2^2\bar{\psi}_4 + 6\bar{\psi}_1\bar{\psi}_2\bar{\psi}_5 + 6\bar{\psi}_1\bar{\psi}_3\bar{\psi}_4 \\ \bar{\psi}_3^3 + 3\bar{\psi}_1\bar{\psi}_4^2 + 3\bar{\psi}_2^2\bar{\psi}_5 + 6\bar{\psi}_1\bar{\psi}_3\bar{\psi}_5 + 6\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4 \\ 3\bar{\psi}_2\bar{\psi}_4^2 + 3\bar{\psi}_3^2\bar{\psi}_4 + 6\bar{\psi}_1\bar{\psi}_4\bar{\psi}_5 + 6\bar{\psi}_2\bar{\psi}_3\bar{\psi}_5 \\ 3\bar{\psi}_1\bar{\psi}_5^2 + 3\bar{\psi}_3\bar{\psi}_4^2 + 3\bar{\psi}_3^2\bar{\psi}_5 + 6\bar{\psi}_2\bar{\psi}_4\bar{\psi}_5 \\ \bar{\psi}_4^3 + 3\bar{\psi}_2\bar{\psi}_5^2 + 6\bar{\psi}_3\bar{\psi}_4\bar{\psi}_5 \\ 3\bar{\psi}_3\bar{\psi}_5^2 + 3\bar{\psi}_4^2\bar{\psi}_5 \\ 3\bar{\psi}_4\bar{\psi}_5^2 \\ \bar{\psi}_5^3 \end{bmatrix}$$

and the wright hand for the matrix equation (46) of **the nonlinear part** becomes:

$$\boldsymbol{C}\left(\sum_{j=0}^{1} \boldsymbol{K}_j \boldsymbol{Q}_{\boldsymbol{\ell}_j} \bar{\Psi}_{\ell_j}^{\alpha,\beta_j}\right) = \left(\boldsymbol{CK}_0\boldsymbol{Q}_2\bar{\Psi}_2^{0.2,0.2} + \boldsymbol{CK}_1\boldsymbol{Q}_3\bar{\Psi}_3^{0.2,0.1}\right) = [\Upsilon_1 \quad \Upsilon_2 \quad \Upsilon_3 \quad \Upsilon_4 \quad \Upsilon_5 \quad \Upsilon_6]^T$$

Where

$$\Upsilon_1 = \left\{ \begin{matrix} 0.893\psi_1^3 + 2.462\psi_1^2\psi_2 + 2.200\psi_1^2\psi_3 + 1.919\psi_1^2\psi_4 + 1.641\psi_1^2\psi_5 + 0.333\psi_1^2 + \\ 2.302\psi_1\psi_2^2 + 4.173\psi_1\psi_2\psi_3 + 3.682\psi_1\psi_2\psi_4 + 3.176\psi_1\psi_2\psi_5 + 0.680\psi_1\psi_2 + 1.911\psi_1\psi_3^2 + \\ 3.405\psi_1\psi_3\psi_4 + 2.959\psi_1\psi_3\psi_5 + 0.662\psi_1\psi_3 + 1.527\psi_1\psi_4^2 + 2.672\psi_1\psi_4\psi_5 + 0.621\psi_1\psi_4 + \\ 1.175\psi_1\psi_5^2 + 0.565\psi_1\psi_5 + 0.728\psi_2^3 + 2.001\psi_2^2\psi_3 + 1.782\psi_2^2\psi_4 + 1.549\psi_2^2\psi_5 + \\ 0.348\psi_2^2 + 1.850\psi_2\psi_3^2 + 3.320\psi_2\psi_3\psi_4 + 2.905\psi_2\psi_3\psi_5 + 0.681\psi_2\psi_5 + 1.499\psi_2\psi_4^2 + \\ 2.637\psi_2\psi_4\psi_5 + 0.641\,\psi_2\psi_4 + 1.165\psi_2\psi_5^2 + 0.584\psi_2\psi_5 + 0.574\psi_3^3 + 1.556\psi_3^2\psi_4 + \\ 1.556\psi_3^2\psi_4 + 1.369\psi_3^2\psi_5 + 0.334\psi_3^2 + 1.413\psi_3\psi_4^2 + 2.499\psi_3\psi_4\psi_5 + 0.630\psi_3\psi_4 + \\ 1.108\psi_3\psi_5^2 + 0.576\psi_3\psi_5 + 0.429\psi_4^3 + 1.144\psi_4^2\psi_5 + 0.298\psi_4^2 + 1.019\psi_4\psi_5^2 + \\ 0.546\psi_4\psi_5 + 0.303\psi_5^3 + 0.250\psi_5^2 \end{matrix} \right\}$$

$$\Upsilon_2 = \left\{ \begin{matrix} 0.994\psi_1^3 + 2.757\psi_1^2\psi_2 + 2.477\psi_1^2\psi_3 + 2.171\psi_1^2\psi_4 + 1.863\psi_1^2\psi_5 - 0.066\psi_1^2 + \\ 2.592\psi_1\psi_2^2 + 4.720\psi_1\psi_2\psi_3 + 4.181\,\psi_1\psi_2\psi_4 + 3.619\psi_1\psi_2\psi_5 - 0.045\psi_1\psi_2 + 2.170\psi_1\psi_3^2 + \\ 3.879\psi_1\psi_3\psi_4 + 3.382\psi_1\psi_3\psi_5 + 0.018\psi_1\psi_3 + 1.746\psi_1\psi_4^2 + 3.063\psi_1\psi_4\psi_5 + 0.062\psi_1\psi_4 + \\ 1.350\psi_1\psi_5^2 + 0.087\psi_1\psi_5 + 0.823\psi_2^3 + 2.272\psi_2^2\psi_3 + 2.030\psi_2^2\psi_4 + 1.770\psi_2^2\psi_5 + \\ 0.009\psi_2^2 + 2.108\psi_2\psi_3^2 + 3.795\psi_2\psi_3\psi_4 + 3.329\psi_2\psi_3\psi_4 + 0.068\psi_2\psi_3 + 1.718\psi_2\psi_4^2 + \\ 3.030\psi_2\psi_4\psi_5 + 0.099\psi_2\psi_5 + 1.342\psi_2\psi_5^2 + 0.116\psi_2\psi_5 + 0.656\psi_3^3 + 1.78419\psi_3^2\psi_4 + \\ 1.573\psi_3^2\psi_5 + 0.051\psi_3^2 + 1.624\psi_3\psi_4^2 + 2.877\psi_3\psi_4\psi_5 + 0.126\psi_3\psi_4 + 1.279\psi_3\psi_5^2 + \\ 0.136\psi_3\psi_5 + 0.495\psi_4^3 + 1.320\psi_4^2\psi_5 + 0.070\psi_4^2 + 1.178\psi_4\psi_5^2 + 0.145\psi_4\psi_5 + \\ 0.351\psi_5^3 + 0.073\psi_5^2 \end{matrix} \right\}$$

$$\Upsilon_3 = \left\{ \begin{matrix} 1.095\psi_1^3 + 3.053\psi_1^2\psi_2 + 2.754\psi_1^2\psi_3 + 2.423\psi_1^2\psi_4 + 2.085\psi_1^2\psi_5 - 0.466\psi_1^2 + \\ 2.883\psi_1\psi_2^2 + 5.267\psi_1\psi_2\psi_3 + 4.680\psi_1\psi_2\psi_4 + 4.062\psi_1\psi_2\psi_5 - 0.771\psi_1\psi_2 + 2.43\psi_1\psi_3^2 + \\ 4.354\psi_1\psi_3\psi_4 + 3.805\psi_1\psi_3\psi_5 - 0.625\psi_1\psi_3 + 1.964\psi_1\psi_4^2 + 3.453\psi_1\psi_4\psi_5 - 0.497\psi_1\psi_5 + \\ 1.525\psi_1\psi_5^2 - 0.389\psi_1\psi_5 + 0.918\psi_2^3 + 2.543\psi_2^2\psi_3 + 2.279\psi_2^2\psi_4 + 1.991\psi_2^2\psi_5 - \\ 0.328\psi_2^2 + 2.366\psi_2\psi_3^2 + 4.269\psi_2\psi_3\psi_4 + 3.753\psi_2\psi_3\psi_5 - 0.545\psi_2\psi_3 + 1.93723\psi_2\psi_4^2 + \\ 3.422\psi_2\psi_4\psi_5 - 0.441\psi_2\psi_4 + 1.518\psi_2\psi_5^2 - 0.350\psi_2\psi_5 + 0.738\psi_3^3 + 2.011\psi_3^2\psi_4 + \\ 1.776\psi_3^2\psi_5 - 0.230\psi_3^2 + 1.834\psi_3\psi_4^2 + 3.254\psi_3\psi_4\psi_5 - 0.378\psi_3\psi_4 + 1.449\psi_3\psi_5^2 - \\ 0.303\psi_3\psi_5 + 0.560\psi_4^3 + 1.496\psi_4^2\psi_5 - 0.157\psi_4^2 + 1.337\psi_4\psi_5^2 - 0.254\psi_4\psi_5 + \\ 0.399\psi_5^3 - 0.104\psi_5^2 \end{matrix} \right\}$$

$$\Upsilon_4 = \left\{ \begin{matrix} 1.196\psi_1^3 + 3.348\psi_1^2\psi_2 + 3.031\psi_1^2\psi_3 + 2.674\psi_1^2\psi_4 + 2.308\psi_1^2\psi_5 - 0.866\psi_1^2 + \\ 3.173\psi_1\psi_2^2 + 5.814\psi_1\psi_2\psi_3 + 5.179\psi_1\psi_2\psi_4 + 4.505\psi_1\psi_2\psi_5 - 1.497\psi_1\psi_2 + 2.689\psi_1\psi_3^2 + \\ 4.829\psi_1\psi_3\psi_4 + 4.228\psi_1\psi_3\psi_5 - 1.269\psi_1\psi_3 + 2.182\psi_1\psi_4^2 + 3.843\psi_1\psi_4\psi_5 - 1.057\psi_1\psi_4 + \\ 1.700\psi_1\psi_5^2 - 0.866\psi_1\psi_5 + 1.014\psi_2^3 + 2.814\psi_2^2\psi_3 + 2.527\psi_2^2\psi_4 + 2.212\psi_2^2\psi_5 - \\ 0.667\psi_2^2 + 2.624\psi_2\psi_3^2 + 4.744\psi_2\psi_3\psi_4 + 4.176\psi_2\psi_3\psi_5 - 1.159\psi_2\psi_3 + 2.156\psi_2\psi_4^2 + \\ 3.814\psi_2\psi_4\psi_5 - 0.983\psi_2\psi_4 + 1.694\psi_2\psi_5^2 - 0.818\psi_2\psi_5 + 0.820\psi_3^3 + 2.238\psi_3^2\psi_4 + \\ 1.980\psi_3^2\psi_5 - 0.512\psi_3^2 + 2.044\psi_3\psi_4^2 + 3.632\psi_3\psi_4\psi_5 + 0.882\psi_3\psi_4 + 1.619\psi_3\psi_5^2 - \\ 0.743\psi_3\psi_5 + 0.625\psi_4^3 + 1.672\psi_4^2\psi_5 - 0.384\psi_4^2 + 1.495\psi_4\psi_5^2 - 0.655\psi_4\psi_5 + \\ 0.447\psi_5^3 - 0.281\psi_5^2 \end{matrix} \right\}$$

$$\Upsilon_5 = \left\{ \begin{matrix} 1.297\psi_1^3 + 3.644\psi_1^2\psi_2 + 3.308\psi_1^2\psi_3 + 2.926\psi_1^2\psi_4 + 2.530\psi_1^2\psi_5 - 1.266\psi_1^2 + \\ 3.463\psi_1\psi_2^2 + 6.360\psi_1\psi_2\psi_3 + 5.678\psi_1\psi_2\psi_4 + 4.948\psi_1\psi_2\psi_5 - 2.223\psi_1\psi_2 + 2.948\psi_1\psi_3^2 + \\ 5.3039\psi_1\psi_3\psi_4 + 4.651\psi_1\psi_3\psi_5 - 1.913\psi_1\psi_3 + 2.400\psi_1\psi_4^2 + 4.233\psi_1\psi_4\psi_5 - 1.616\psi_1\psi_4 + \\ 1.874\psi_1\psi_5^2 - 1.343\psi_1\psi_5 + 1.109\psi_2^3 + 3.085\psi_2^2\psi_3 + 2.775\psi_2^2\psi_4 + 2.434\psi_2^2\psi_5 - \\ 1.006\psi_2^2 + 2.882\psi_2\psi_3^2 + 5.218\psi_2\psi_3\psi_4 + 4.600\psi_2\psi_3\psi_5 - 1.773\psi_2\psi_3 + 2.374\psi_2\psi_4^2 + \\ 4.207\psi_2\psi_4\psi_5 - 1.525\psi_2\psi_4 + 1.870\psi_2\psi_5^2 - 1.286\psi_2\psi_5 + 0.903\psi_3^3 + 2.465\psi_3^2\psi_4 + \\ 2.184\psi_3^2\psi_5 - 0.794\psi_3^2 + 2.254\psi_3\psi_4^2 + 4.010\psi_3\psi_4\psi_5 - 1.387\psi_3\psi_4 + 1.789\psi_3\psi_5^2 - \\ 1.183\psi_3\psi_5 + 0.690\psi_4^3 + 1.847\psi_4^2\psi_5 - 0.612\psi_4^2 + 1.654\psi_4\psi_5^2 - 1.055\psi_4\psi_5 + \\ 0.495\psi_5^3 - 0.458\psi_5^2 \end{matrix} \right\}$$

$$\Upsilon_6 = \left\{ \begin{matrix} 1.398\psi_1^3 + 3.939\psi_1^2\psi_2 + 3.585\psi_1^2\psi_3 + 3.177\psi_1^2\psi_4 + 2.752\psi_1^2\psi_5 - 1.666\psi_1^2 + \\ 3.753\psi_1\psi_2^2 + 6.907\psi_1\psi_2\psi_3 + 6.177\psi_1\psi_2\psi_4 + 5.390\psi_1\psi_2\psi_5 - 2.949\psi_1\psi_2 + 3.207\psi_1\psi_3^2 + \\ 5.778\psi_1\psi_3\psi_4 + 5.073\psi_1\psi_3\psi_5 - 2.557\psi_1\psi_3 + 2.6190\psi_1\psi_4^2 + 4.623\psi_1\psi_4\psi_5 - 2.1761\psi_1\psi_4 + \\ 2.049\psi_1\psi_5^2 - 1.820\psi_1\psi_5 + 1.205\psi_2^3 + 3.357\psi_2^2\psi_3 + 3.024\psi_2^2\psi_4 + 2.655\psi_2^2\psi_5 - \\ 1.345\psi_2^2 + 3.140\psi_2\psi_3^2 + 5.692\psi_2\psi_3\psi_4 + 5.024\psi_2\psi_3\psi_5 - 2.386\psi_2\psi_3 + 2.593\psi_2\psi_4^2 + \\ 4.599\psi_2\psi_4\psi_5 - 2.067\psi_2\psi_4 + 2.047\psi_2\psi_5^2 - 1.754\psi_2\psi_5 + 0.985\psi_3^3 + 2.693\psi_3^2\psi_4 + \\ 2.387\psi_3^2\psi_5 - 1.077\psi_3^2 + 2.465\psi_3\psi_4^2 + 4.388\psi_3\psi_4\psi_5 - 1.892\psi_3\psi_4 + 1.960\psi_3\psi_5^2 - \\ 1.623\psi_3\psi_5 + 0.755\psi_4^3 + 2.023\psi_4^2\psi_5 - 0.840\psi_4^2 + 1.813\psi_4\psi_5^2 - 1.456\psi_4\psi_5 + \\ 0.542\psi_5^3 - 0.636\psi_5^2 \end{matrix} \right\}$$

Also, we have the matrix representation of condition form (49):

$$[U_0 : C_0] = [0 \quad -1.127060 \quad -1.119174 \quad -1.073671 \quad -1.0 \quad -1.089124 \quad : \quad -2]$$

Substituting the above matrices for fundamental equation, the augmented matrix is obtained based on condition which is:

0	2	0	1	0	0	;
0.19867	2.15682	1.69637	2.26876	1.70044	1.22434	;
0.38942	2.35311	2.11773	2.81394	2.39934	1.96867	;
0.56464	2.55524	2.48548	3.30266	3.03369	2.68455	;
0.71736	2.74719	2.82264	3.76388	3.64373	3.40098	;
0.84147	2.91647	3.12664	4.19587	4.23094	4.11587	;

0.33333	0.3125	0.29412	0.27778	0.26316	0.25	0.23809	0.22727	0.21739	;
−0.06667	−0.02083	0.00840	0.02778	0.04094	0.05	0.05628	0.06061	0.06355	;
−0.46667	−0.35417	−0.27731	−0.22222	−0.18129	−0.15	−0.12554	−0.10606	−0.09030	;
−0.86667	−0.6875	−0.56303	−0.47222	−0.40351	−0.35	−0.30736	−0.27273	−0.24415	;
−1.26667	−1.02083	−0.84874	−0.72222	−0.62573	−0.55	−0.48918	−0.43939	−0.39799	;
−1.66667	−1.35417	−1.13445	−0.97222	−0.84795	−0.75	−0.67099	−0.60606	−0.55184	;

0.76923	0.66667	0.58824	0.52632	0.47619	0.43478	0.40000	0.37037	0.34483	0.32258	0.30303	0.28571	0.27027	;
0.85619	0.74667	0.66231	0.59528	0.54071	0.49539	0.45714	0.42442	0.39611	0.37136	0.34954	0.33016	0.31282	;
0.94314	0.82667	0.73638	0.66425	0.60522	0.55600	0.51429	0.47848	0.44739	0.42014	0.39605	0.37460	0.35538	;
1.03010	0.90667	0.81046	0.73321	0.66974	0.61660	0.57143	0.53253	0.49867	0.46892	0.44257	0.41905	0.39793	;
1.11706	0.98667	0.88453	0.80218	0.73426	0.67721	0.62857	0.58659	0.54996	0.51770	0.48908	0.46349	0.44048	;
1.20401	1.06667	0.95861	0.87114	0.79877	0.73781	0.68571	0.64064	0.60124	0.56648	0.53559	0.50798	0.48304	;

$$\begin{bmatrix} -3.4328225456022 \\ -0.4587462830933 \\ 1.54741534196750 \\ 3.48113206807783 \\ 5.39344330115713 \\ 7.27407661372808 \end{bmatrix}$$

Solving this nonlinear system with the matrix representation of condition, apply Newton-Raphson iterative method, we obtain the coefficients of the generalized Taylor series:

$$\Psi = [1 \quad -4.309e-012 \quad 3.147e-012 \quad 1.156e-011 \quad -1.531e-011 \quad 2.0]^T$$

Substituting the elements $\psi_k(k=\overline{0:5})$ for truncated equation (8), we can get the approximate solution $\tilde{y}(t)$ of $y(t)$:

$$\begin{aligned}\tilde{y}(t) = 1.0 + 2.000000000005163\, t \\ - 0.000000000004693\, t^{0.2} \\ + 0.000000000003547\, t^{0.4} \\ + 0.000000000012944\, t^{0.6} \\ - 0.000000000016447\, t^{0.8} \\ \cong 1.0 + 2.0t\end{aligned}$$

Which is the exact solution of the equation.

Example (2):

Consider we have a high-order nonlinear IFDEs of F-H Type on fractional order lies in the interval (1,2) and for all $t \in [0,1]$:

$$\begin{aligned}{}_0^C D_t^{1.6} y(t) - t^2\, {}_0^C D_t^{0.8} y(t) + \cosh(t)\, y(t) \\ = f(t) \\ + \int_0^1 \{(s-e^t)[{}_0^C D_s^{0.5} y(s)]^4\} ds\end{aligned}$$

and

$$\begin{aligned}f(t) = \frac{24}{\Gamma(3.4)} t^{2.4} - \frac{24}{\Gamma(4.2)} t^{5.2} \\ + \cosh(t)\,(t^4+1) \\ + \left[\frac{24}{\Gamma(4.5)}\right]^4 \left(\frac{e^t}{15} - \frac{1}{16}\right)\end{aligned}$$

with the two boundary conditions:

(I) $y(0) = 1$ and $y(1) = 2$.

(II) $y(0) = 1$ and $y^{(4)}(1) = 24$.

While the exact solution is $y(t) = 1 + t^4$.

By apply the algorithm [AGTM-NLFH], for ($N = 5$ and $N_* = 4$), obtain the fundamental matrix relation after running the general MatLab program Taylor Main, we have ($\ell_0 = 4$):

$$[\mathcal{P}_0 \mathcal{C}^\alpha \bar{\mathcal{M}}_2 + \mathcal{P}_1 \mathcal{C}^\alpha \bar{\mathcal{M}}_1 + \mathcal{P}_2 \mathcal{C}^\alpha \bar{\mathcal{M}}_0]\boldsymbol{\Psi} - [\boldsymbol{\mathcal{C}} \boldsymbol{K}_0 \boldsymbol{Q}_{\ell_0} \bar{\boldsymbol{\Psi}}_{\ell_0}^{0.8,0.5}] = \mathcal{F}$$

Where **the linear part** is equal to:

$$[\mathcal{P}_0\mathcal{C}^{\alpha}\overline{\mathcal{M}}_2+\mathcal{P}_1\mathcal{C}^{\alpha}\overline{\mathcal{M}}_1+\mathcal{P}_2\mathcal{C}^{\alpha}\overline{\mathcal{M}}_0]\boldsymbol{\Psi}=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0\\ 1.020066 & 0.262220 & 1.042480 & 0.301334 & 0.053743 & 0.007086\\ 1.081072 & 0.397665 & 1.092018 & 0.530227 & 0.162937 & 0.037254\\ 1.185465 & 0.485826 & 1.109332 & 0.718989 & 0.303268 & 0.095790\\ 1.337434 & 0.561201 & 1.079819 & 0.847481 & 0.448230 & 0.178771\\ 1.543080 & 0.656761 & 1.005689 & 0.891789 & 0.562985 & 0.270808 \end{bmatrix}\begin{bmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\\ \psi_4\\ \psi_5\end{bmatrix}$$

And **the non-linear part** is equal to:

$$\boldsymbol{\mathcal{C}K}_0\boldsymbol{Q}_4\overline{\boldsymbol{\Psi}}_4^{0.8,0.5}=[\Upsilon_1\quad \Upsilon_2\quad \Upsilon_3\quad \Upsilon_4\quad \Upsilon_5\quad \Upsilon_6]^T$$

Where

$$\Upsilon_1=\left\{\begin{matrix} -0.218\psi_1^4-0.440\psi_1^3\psi_2-0.166\psi_1^3\psi_3-0.051\psi_1^3\psi_4-0.013\psi_1^3\psi_5-0.372\psi_1^2\psi_2^2-\\ 0.302\psi_1^2\psi_2\psi_3-0.098\psi_1^2\psi_2\psi_4-0.027\psi_1^2\psi_2\psi_5-0.064\psi_1^2\psi_3^2-0.043\psi_1^2\psi_3\psi_4-0.012\psi_1^2\psi_3\psi_5-\\ 0.007\psi_1^2\psi_4^2-0.004\psi_1^2\psi_4\psi_5-0.0006\psi_1^2\psi_5^2-0.150\psi_1\psi_2^3-0.193\psi_1\psi_2^3\psi_3-0.065\psi_1\psi_2^3\psi_4-\\ 0.018\psi_1\psi_2^2\psi_5-0.085\psi_1\psi_2\psi_3^2-0.059\psi_1\psi_2\psi_3\psi_4-0.01\psi_1\psi_2\psi_3\psi_5-0.010\psi_1\psi_2\psi_4^2-0.006\psi_1\psi_2\psi_4\psi_5-\\ 0.0009\psi_1\psi_2\psi_5^2-0.013\psi_1\psi_3^3-0.013\psi_1\psi_3^2\psi_4-0.004\psi_1\psi_3^2\psi_5-0.005\psi_1\psi_3\psi_5^2-0.003\psi_1\psi_3\psi_4\psi_5-\\ 0.0004\psi_1\psi_3\psi_5^2-0.0006\psi_1\psi_4^3-0.0005\psi_1\psi_4^2\psi_5-0.0001\psi_1\psi_4\psi_5^2-0.00001\psi_1\psi_5^3-0.024\psi_2^4-\\ 0.042\psi_2^3\psi_3-0.01\psi_2^3\psi_4-0.004\psi_2^3\psi_5-0.029\psi_2^2\psi_3^2-0.020\psi_2^2\psi_3\psi_4-0.006\psi_2^2\psi_3\psi_5-\\ 0.003\psi_2^2\psi_4^2-0.002\psi_2^2\psi_4\psi_5-0.0003\psi_2^2\psi_5^2-0.009\psi_2\psi_3^3-0.009\psi_2\psi_3^2\psi_4-0.003\psi_2\psi_3^2\psi_5-\\ 0.003\psi_2\psi_3\psi_4^2-0.002\psi_2\psi_3\psi_1^2\psi_5-0.0003\psi_2\psi_3\psi_5^2-0.0004\psi_2\psi_4^3-0.0004\psi_2\psi_4^2\psi_5-0.0001\psi_2\psi_4\psi_5^2-\\ 0.00001\psi_2\psi_5^2-0.001\psi_3^4-0.001\psi_3^3\psi_4-0.0004\psi_3^3\psi_5-0.0009\psi_3^2\psi_4^2-0.0005\psi_3^2\psi_4\psi_5-\\ 0.00008\psi_3^2\psi_5^2-0.0002\psi_3\psi_4^3-0.0002\psi_3\psi_4^2\psi_5-0.00006\psi_3\psi_4\psi_5^2-0.000007\psi_3\psi_5^3-0.00002\psi_5^4-\\ 0.00002\psi_4^3\psi_5-0.00001\psi_4^2\psi_5^2-0.000002\psi_4\psi_5^3-2.2e-7\psi_5^4 \end{matrix}\right\}$$

$$\Upsilon_2=\left\{\begin{matrix} -0.374\psi_1^4-0.830\psi_1^3\psi_2-0.342\psi_1^3\psi_3-0.115\psi_1^3\psi_4-0.033\psi_1^3\psi_5-0.769\psi_1^2\psi_2^2-\\ 0.677\psi_1^2\psi_2\psi_3-0.238\psi_1^2\psi_2\psi_4-0.071\psi_1^2\psi_2\psi_5-0.156\psi_1^2\psi_3^2-0.113\psi_1^2\psi_3\psi_4-0.034\psi_1^2\psi_3\psi_5-\\ 0.021\psi_1^2\psi_4^2-0.013\psi_1^2\psi_4\psi_5-0.002\psi_1^2\psi_5^2-0.338\psi_1\psi_2^3-0.467\psi_1\psi_2^3\psi_3-0.170\psi_1\psi_2^3\psi_4-\\ 0.051\psi_1\psi_2^2\psi_5-0.051\psi_1\psi_2\psi_3^2-0.222\psi_1\psi_2\psi_3\psi_4-0.165\psi_1\psi_2\psi_3\psi_5-0.051\psi_1\psi_2\psi_4^2-0.031\psi_1\psi_2\psi_4\psi_5-\\ 0.019\psi_1\psi_2\psi_5^2-0.003\psi_1\psi_3^3-0.036\psi_1\psi_3^2\psi_4-0.041\psi_1\psi_3^2\psi_5-0.013\psi_1\psi_3\psi_5^2-0.015\psi_1\psi_3\psi_4\psi_5-\\ 0.010\psi_1\psi_3\psi_5^2-0.001\psi_1\psi_4^3-0.002\psi_1\psi_4^2\psi_5-0.002\psi_1\psi_4\psi_5^2-0.00007\psi_1\psi_5^3-0.058\psi_2^4-\\ 0.110\psi_2^3\psi_3-0.041\psi_2^3\psi_4-0.012\psi_2^3\psi_5-0.081\psi_2^2\psi_3^2-0.061\psi_2^2\psi_3\psi_4-0.019\psi_2^2\psi_3\psi_5-\\ 0.011\psi_2^2\psi_4^2-0.007\psi_2^2\psi_4\psi_5-0.001\psi_2^2\psi_5^2-0.026\psi_2\psi_3^3-0.031\psi_2\psi_3^2\psi_4-0.009\psi_2\psi_3^2\psi_5-\\ 0.012\psi_2\psi_3\psi_4^2-0.007\psi_2\psi_3\psi_1^2\psi_5-0.001\psi_2\psi_3\psi_5^2-0.001\psi_2\psi_4^3-0.001\psi_2\psi_4^2\psi_5-0.0005\psi_2\psi_4\psi_5^2-\\ 0.00005\psi_2\psi_5^2-0.003\psi_3^4-0.005\psi_3^3\psi_4-0.001\psi_3^3\psi_5-0.003\psi_3^2\psi_4^2-0.002\psi_3^2\psi_4\psi_5-\\ 0.0003\psi_3^2\psi_5^2-0.0008\psi_3\psi_4^3-0.0008\psi_3\psi_4^2\psi_5-0.0002\psi_3\psi_4\psi_5^2-0.00003\psi_3\psi_5^3-0.00008\psi_5^4-\\ 0.0001\psi_4^3\psi_5-0.00005\psi_4^2\psi_5^2-0.00001\psi_4\psi_5^3-0.000001\psi_5^4 \end{matrix}\right\}$$

$$\Upsilon_3=\left\{\begin{matrix} 0.563\psi_1^4-1.307\psi_1^3\psi_2-0.557\psi_1^3\psi_3-0.193\psi_1^3\psi_4-0.057\psi_1^3\psi_5-1.253\psi_1^2\psi_2^2-\\ 1.135\psi_1^2\psi_2\psi_3-0.409\psi_1^2\psi_2\psi_4-0.124\psi_1^2\psi_2\psi_5-0.267\psi_1^2\psi_3^2-0.198\psi_1^2\psi_3\psi_4-0.061\psi_1^2\psi_3\psi_5-\\ 0.037\psi_1^2\psi_4^2-0.023\psi_1^2\psi_4\psi_5-0.003\psi_1^2\psi_5^2-0.566\psi_1\psi_2^3-0.801\psi_1\psi_2^3\psi_3-0.297\psi_1\psi_2^3\psi_4-\\ 0.092\psi_1\psi_2^2\psi_5-0.389\psi_1\psi_2\psi_3^2-0.295\psi_1\psi_2\psi_3\psi_4-0.093\psi_1\psi_2\psi_3\psi_5-0.057\psi_1\psi_2\psi 4^2-0.036\psi_1\psi_2\psi_4\psi_5-\\ 0.005\psi_1\psi_2\psi_5^2-0.064\psi_1\psi_3^3-0.074\psi_1\psi_3^2\psi_4-0.023\psi_1\psi_3^2\psi_5-0.029\psi_1\psi_3\psi_5^2-0.018\psi_1\psi_3\psi_4\psi_5-\\ 0.003\psi_1\psi_3\psi_5^2-0.003\psi_1\psi_4^3-0.003\psi_1\psi_4^2\psi_5-0.001\psi_1\psi_4\psi_5^2-0.0001\psi_1\psi_5^3-0.1000\psi_2^4-\\ 0.194\psi_2^3\psi_3-0.073\psi_2^3\psi_4-0.023\psi_2^3\psi_5-0.144\psi_2^2\psi_3^2-0.111\psi_2^2\psi_3\psi_4-0.035\psi_2^2\psi_3\psi_5-\\ 0.021\psi_2^2\psi_4^2-0.014\psi_2^2\psi_4\psi_5-0.002\psi_2^2\psi_5^2-0.048\psi_2\psi_3^3-0.057\psi_2\psi_3^2\psi_4-0.018\psi_2\psi_3^2\psi_5-\\ 0.022\psi_2\psi_3\psi_4^2-0.014\psi_2\psi_3\psi_1^2\psi_5-0.002\psi_2\psi_3\psi_5^2-0.003\psi_2\psi_4^3-0.002\psi_2\psi_4^2\psi_5-0.0009\psi_2\psi_4\psi_5^2-\\ 0.0001\psi_2\psi_5^2-0.006\psi_3^4-0.009\psi_3^3\psi_4-0.003\psi_3^3\psi_5-0.005\psi_3^2\psi_4^2-0.003\psi_3^2\psi_4\psi_5-\\ 0.0006\psi_3^2\psi_5^2-0.001\psi_3\psi_4^3-0.001\psi_3\psi_4^2\psi_5-0.0005\psi_3\psi_4\psi_5^2-0.00005\psi_3\psi_5^3-0.0001\psi_5^4-\\ 0.0002\psi_4^3\psi_5-0.0001\psi_4^2\psi_5^2-0.00002\psi_4\psi_5^3-0.000002\psi_5^4 \end{matrix}\right\}$$

$$\Upsilon_4=\left\{\begin{matrix} -0.794\psi_1^4-1.888\psi_1^3\psi_2-0.820\psi_1^3\psi_3-0.288\psi_1^3\psi_4-0.086\psi_1^3\psi_5-1.843\psi_1^2\psi_2^2-\\ 1.693\psi_1^2\psi_2\psi_3-0.618\psi_1^2\psi_2\psi_4-0.189\psi_1^2\psi_2\psi_5-0.403\psi_1^2\psi_3^2-0.302\psi_1^2\psi_3\psi_4-0.094\psi_1^2\psi_3\psi_5-\\ 0.057\psi_1^2\psi_4^2-0.036\psi_1^2\psi_4\psi_5-0.005\psi_1^2\psi_5^2-0.845\psi_1\psi_2^3-1.209\psi_1\psi_2^3\psi_3-0.453\psi_1\psi_2^3\psi_4-\\ 0.141\psi_1\psi_2^2\psi_5-0.592\psi_1\psi_2\psi_3^2-0.453\psi_1\psi_2\psi_3\psi_4-0.144\psi_1\psi_2\psi_3\psi_5-0.088\psi_1\psi_2\psi_4^2-0.056\psi_1\psi_2\psi_4\psi_5-\\ 0.009\psi_1\psi_2\psi_5^2-0.098\psi_1\psi_3^3-0.115\psi_1\psi_3^2\psi_4-0.037\psi_1\psi_3^2\psi_5-0.045\psi_1\psi_3\psi_5^2-0.029\psi_1\psi_3\psi_4\psi_5-\\ 0.004\psi_1\psi_3\psi_5^2-0.005\psi_1\psi_4^3-0.005\psi_1\psi_4^2\psi_5-0.001\psi_1\psi_4\psi_5^2-0.0002\psi_1\psi_5^3-0.150\psi_2^4-\\ 0.295\psi_2^3\psi_3-0.113\psi_2^3\psi_4-0.035\psi_2^3\psi_5-0.221\psi_2^2\psi_3^2-0.172\psi_2^2\psi_3\psi_4-0.055\psi_2^2\psi_3\psi_5-\\ 0.033\psi_2^2\psi_4^2-0.022\psi_2^2\psi_4\psi_5-0.003\psi_2^2\psi_5^2-0.075\psi_2\psi_3^3-0.088\psi_2\psi_3^2\psi_4-0.028\psi_2\psi_3^2\psi_5-\\ 0.035\psi_2\psi_3\psi_4^2-0.023\psi_2\psi_3\psi_1^2\psi_5-0.003\psi_2\psi_3\psi_5^2-0.004\psi_2\psi_4^3-0.004\psi_2\psi_4^2\psi_5-0.001\psi_2\psi_4\psi_5^2-\\ 0.0001\psi_2\psi_5^2-0.009\psi_3^4-0.015\psi_3^3\psi_4-0.005\psi_3^3\psi_5-0.009\psi_3^2\psi_4^2-0.006\psi_3^2\psi_4\psi_5-\\ 0.001\psi_3^2\psi_5^2-0.002\psi_3\psi_4^3-0.002\psi_3\psi_4^2\psi_5-0.0008\psi_3\psi_4\psi_5^2-0.00009\psi_3\psi_5^3-0.0002\psi_5^4-\\ 0.0003\psi_4^3\psi_5-0.0001\psi_4^2\psi_5^2-0.00003\psi_4\psi_5^3-0.000003\psi_5^4 \end{matrix}\right\}$$

$$\varUpsilon_5=\left\{\begin{array}{c} -1.075\psi_1^4-\ 2.595\psi_1^3\psi_2-1.140\psi_1^3\psi_3\ -\ 0.404\psi_1^3\psi_4\ -0.121\psi_1^3\psi_5\ -\ 2.561\psi_1^2\psi_2^2- \\ 2.372\psi_1^2\psi_2\psi_3-\ \ 0.871\psi_1^2\psi_2\psi_4-\ 0.268\psi_1^2\psi_2\psi_5-0.569\psi_1^2\psi_3^2-\ 0.429\psi_1^2\psi_3\psi_4-\ \ 0.134\psi_1^2\psi_3\psi_5- \\ 0.082\psi_1^2\psi_4^2-0.052\psi_1^2\psi_4\psi_5-\ 0.008\psi_1^2\psi_5^2\ -\ 1.184\psi_1\psi_2^3\ -1.705\psi_1\psi_2^3\psi_3\ -0.642\psi_1\psi_2^3\psi_4- \\ 0.202\psi_1\psi_2^2\psi_5\ -0.840\psi_1\psi_2\psi_3^2-0.645\psi_1\psi_2\psi_3\psi_4\ -0.205\psi_1\psi_2\psi_3\psi_5\ -\ \ 0.125\psi_1\psi_2\psi_4^2-\ \ 0.081\psi_1\psi_2\psi_4\psi_5\ - \\ 0.013\psi_1\psi_2\psi_5^2\ -0.140\psi_1\psi_3^3-\ 0.164\psi_1\psi_3^2\psi_4\ -0.053\psi_1\psi_3^2\psi_5-\ \ 0.064\psi_1\psi_3\psi_5^2\ -\ \ 0.042\psi_1\psi_3\psi_4\psi_5\ - \\ 0.006\psi_1\psi_3\psi_5^2-\ 0.008\psi_1\psi_4^3-\ 0.008\psi_1\psi_4^2\psi_5-\ \ \ 0.002\psi_1\psi_4\psi_5^2\ -\ \ 0.0003\psi_1\psi_5^3-\ \ 0.212\psi_2^4\ - \\ 0.419\psi_2^3\psi_3-\ 0.161\psi_2^3\psi_4-0.051\psi_2^3\psi_5-\ 0.315\psi_2^2\psi_3^2-\ 0.246\psi_2^2\psi_3\psi_4\ -0.0795\psi_2^2\psi_3\psi_5\ - \\ 0.048\psi_2^2\psi_4^2-0.031\psi_2^2\psi_4\psi_5-0.005\psi_2^2\psi_5^2-\ 0.107\psi_2\psi_3^3-\ 0.127\psi_2\psi_3^2\psi_4\ -\ \ 0.041\psi_2\psi_3^2\psi_5- \\ 0.035\psi_2\psi_3\psi_4^2-0.033\psi_2\psi_3\psi_1^2\psi_5-0.005\psi_2\psi_3\psi_5^2-0.004\psi_2\psi_4^3-0.006\psi_2\psi_4^2\psi_5\ -0.002\psi_2\psi_4\psi_5^2\ - \\ 0.0002\psi_2\psi_5^2-\ 0.013\psi_3^4\ -\ 0.022\psi_3^3\psi_4-\ 0.007\psi_3^3\psi_5-\ \ 0.013\psi_3^2\psi_4^2\ -\ \ 0.008\psi_3^2\psi_4\psi_5- \\ 0.001\psi_3^2\psi_5^2-\ 0.003\psi_3\psi_4^3-0.003\psi_3\psi_4^2\psi_5\ -0.001\psi_3\psi_4\psi_5^2\ -\ 0.0001\psi_3\psi_5^3-0.0003\psi_5^4- \\ 0.0004\psi_4^3\psi_5-0.0002\psi_4^2\psi_5^2\ -0.00005\psi_4\psi_5^3\ -0.000004\psi_5^4 \end{array}\right\}$$

$$\varUpsilon_6=\left\{\begin{array}{c} -1.415\psi_1^4-\ 3.451\psi_1^3\psi_2-\ 1.527\psi_1^3\psi_3\ -0.544\psi_1^3\psi_4\ -0.121\psi_1^3\psi_5\ -\ 2.561\psi_1^2\psi_2^2- \\ 3.195\psi_1^2\psi_2\psi_3-\ \ 1.178\psi_1^2\psi_2\psi_4-0.364\psi_1^2\psi_2\psi_5-0.770\psi_1^2\psi_3^2-\ 0.582\psi_1^2\psi_3\psi_4-\ 0.183\psi_1^2\psi_3\psi_5- \\ 0.112\psi_1^2\psi_4^2-0.071\psi_1^2\psi_4\psi_5-\ 0.011\psi_1^2\psi_5^2\ -\ 1.595\psi_1\psi_2^3\ -2.307\psi_1\psi_2^3\psi_3\ -\ 0.872\psi_1\psi_2^3\psi_4- \\ 0.274\psi_1\psi_2^2\psi_5\ -\ 1.140\psi_1\psi_2\psi_3^2-0.878\psi_1\psi_2\psi_3\psi_4\ -0.280\psi_1\psi_2\psi_3\psi_5\ -\ 0.171\psi_1\psi_2\psi_4^2-\ 0.111\psi_1\psi_2\psi_4\psi_5\ - \\ 0.018\psi_1\psi_2\psi_5^2\ -0.191\psi_1\psi_3^3-0.224\psi_1\psi_3^2\psi_4\ -\ 0.072\psi_1\psi_3^2\psi_5-\ \ 0.088\psi_1\psi_3\psi_5^2\ -\ 0.057\psi_3\psi_4\psi_5\ - \\ 0.009\psi_1\psi_3\psi_5^2-\ 0.011\psi_1\psi_4^3-\ 0.011\psi_1\psi_4^2\psi_5-\ \ 0.003\psi_1\psi_4\psi_5^2\ -\ \ 0.0004\psi_1\psi_5^3-\ \ 0.287\psi_2^4\ - \\ 0.569\psi_2^3\psi_3-0.219\psi_2^3\psi_4-\ 0.070\psi_2^3\psi_5-\ 0.429\psi_2^2\psi_3^2-\ \ 0.335\psi_2^2\psi_3\psi_4\ -\ 0.108\psi_2^2\psi_3\psi_5\ - \\ 0.066\psi_2^2\psi_4^2-\ 0.043\psi_2^2\psi_4\psi_5-\ 0.007\psi_2^2\psi_5^2-0.146\psi_2\psi_3^3-\ 0.173\psi_2\psi_3^2\psi_4\ -\ \ 0.056\psi_2\psi_3^2\psi_5- \\ 0.069\psi_2^3\psi_4^2-0.045\psi_2\psi_3\psi_1^2\psi_5-\ 0.007\psi_2\psi_3\psi_5^2-0.009\psi_2\psi_4^3-\ 0.009\psi_2\psi_4^2\psi_5\ -0.003\psi_2\psi_4\psi_5^2\ - \\ 0.0003\psi_2\psi_5^2-\ 0.018\psi_3^4\ -0.030\psi_3^3\psi_4-\ 0.009\psi_3^3\psi_5-\ 0.018\psi_3^2\psi_4^2\ -\ \ 0.012\psi_3^2\psi_4\psi_5- \\ 0.002\psi_3^2\psi_5^2-\ 0.004\psi_3\psi_4^3-0.004\psi_3\psi_4^2\psi_5\ -\ 0.001\psi_3\psi_4\psi_5^2\ -\ 0.0001\psi_3\psi_5^3-0.0005\psi_5^4- \\ 0.0006\psi_4^3\psi_5-0.0003\psi_4^2\psi_5^2\ -0.00007\psi_4\psi_5^3\ -\ 0.000006\psi_5^4 \end{array}\right\}$$

And $\mathcal{F}$ is a vector of order (1×6) defined in (52) by:

$$\mathcal{F}=[1.075519\quad 1.533179\quad 2.644981\quad 4.553316\quad 7.184296\quad 10.194218]^T$$

Thus

$$[\widetilde{W};\widetilde{V}\colon\widetilde{F}]=\left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 1.020066 & 0.262220 & 1.042480 & 0.301334 & 0.053743 & 0.007086 \\ 1.081072 & 0.397665 & 1.092018 & 0.530227 & 0.162937 & 0.037254 \\ 1.185465 & 0.485826 & 1.109332 & 0.718989 & 0.303268 & 0.095790 \\ 1.337434 & 0.561201 & 1.079819 & 0.847481 & 0.448230 & 0.178771 \\ 1.543080 & 0.656761 & 1.005689 & 0.891789 & 0.562985 & 0.270808 \end{array}\right.\begin{array}{c} ; \\ ; \\ ; \\ ; \\ ; \\ ; \end{array}$$

$$\begin{array}{cccccccc} -0.14204 & -0.08333 & -0.05482 & -0.03881 & -0.02893 & -0.02240 & -0.01785 & -0.01456 \\ -0.24268 & -0.15713 & -0.11308 & -0.08695 & -0.06993 & -0.05811 & -0.04948 & -0.04295 \\ -0.36556 & -0.24724 & -0.18422 & -0.14571 & -0.11999 & -0.10171 & -0.08810 & -0.07761 \\ -0.51540 & -0.35713 & -0.27098 & -0.21738 & -0.18104 & -0.15488 & -0.13520 & -0.11987 \\ -0.69768 & -0.42080 & -0.37650 & -0.30454 & -0.25530 & -0.21956 & -0.19248 & -0.17128 \\ -0.91856 & -0.65277 & -0.50438 & -0.41019 & -0.34529 & -0.29793 & -0.26190 & -0.23358 \end{array}$$

$$\left.\begin{array}{cccccccccccc} -0.01211 & -0.01022 & -0.00875 & -0.00757 & -0.00662 & -0.00583 & -0.00518 & -0.00463 & -0.00416 & ; & 1.075519 \\ -0.03785 & -0.03378 & -0.03045 & -0.02770 & -0.02538 & -0.02340 & -0.02170 & -0.02022 & -0.01892 & ; & 1.533179 \\ -0.06929 & -0.06254 & -0.05696 & -0.05227 & -0.04829 & -0.04486 & -0.04187 & -0.03926 & -0.03694 & ; & 2.644981 \\ -0.10762 & -0.09761 & -0.08928 & -0.08224 & -0.07623 & -0.07102 & -0.06648 & -0.06247 & -0.05892 & ; & 4.553316 \\ -0.15425 & -0.14027 & -0.12859 & -0.11870 & -0.11021 & -0.10285 & -0.09640 & -0.09071 & -0.08566 & ; & 7.184296 \\ -0.21075 & -0.19196 & -0.17623 & -0.16287 & -0.15139 & -0.14141 & -0.13267 & -0.12493 & -0.11805 & ; & 10.19422 \end{array}\right]$$

The matrix of boundary conditions of (I)-type is computed as:

$$[U_0;C_0]=[1\quad 0\quad 0\quad 0\quad 0\quad 0\quad ;\quad 1]$$

$$[U_1;C_1]=[1\quad 1.073671\quad 0.699484\quad 0.335434\quad 0.128920\quad 0.041666\quad ;\quad 2]$$

Also the matrix of boundary conditions of (II)-type is formed as:

$$[U_0;C_0]=[1\quad 0\quad 0\quad 0\quad 0\quad 0\quad ;\quad 1]$$

$$[U_1;C_1]=[0\quad 0\quad 0\quad 0\quad 0\quad 1\quad ;\quad 24]$$

Substituting the above matrices into fundamental matrix equation and the system has the solution after solving it, whenever $N=5$ and $N_*=4$: For type (I) conditions;

$$\Psi=[1.0\quad -0.000120\quad -0.003693\quad 0.025898\quad -0.098450\quad 24.161213]^T$$

Therefore, we find the approximation solution $\tilde{y}_I(t)\ to\ y(t)$:

$$\tilde{y}_I(t) = 1.0 + 1.00671722128179\, t^4 - 0.000128927743466\, t^{4/5} - 0.002583294854915\, t^{8/5} + 0.008687381686886\, t^{12/5} - 0.012692380370287\, t^{16/5}$$

For type (II) conditions;

$$\Psi = [1.0 \quad 0.017295 \quad -0.002152 \quad -0.021499 \quad -0.010604 \quad 24.0]^T$$

Therefore, we find the approximation solution $\tilde{y}_{II}(t)$ *to* $y(t)$:

$$\tilde{y}_{II}(t) = 1.0 + t^4 + 0.01856873616356\, t^{4/5} - 0.0015049904947288 27\, t^{8/5} - 0.00721134391618\, t^{12/5} - 0.001367118312978\, t^{16/5}$$

The result in Tables (1 and 2) shows the least square errors and running time for boundary conditions in type (I and II), respectively with different values of N and N_*.

Table (1): Comparison results for different value of N and N_* for boundary conditions in type I

t	Exact	Present Method for $N = 5$			
		$N_* = 4$	$N_* = 7$	$N_* = 10$	$N_* = 15$
0.0	1.0	1.0	1.0	1.0	1.0
0.1	1.0001	1.0000419	1.0000999	1.0000999	1.0001
0.2	1.0016	1.0014874	1.0015998	1.0015999	1.0016
0.3	1.0081	1.0079425	1.0080997	1.0080999	1.0081
0.4	1.0256	1.0254008	1.0255997	1.0255999	1.0256
0.5	1.0625	1.0622583	1.0624996	1.0624999	1.0625
0.6	1.1296	1.1293182	1.1295996	1.1295999	1.1296
0.7	1.2401	1.2397930	1.2400995	1.2400999	1.2401
0.8	1.4096	1.4093061	1.4095995	1.4095999	1.4096
0.9	1.6561	1.6558922	1.6560996	1.6560999	1.6561
1.0	2.0	2.0	2.0	2.0	2.0
L.S.E		$4.42187e-006$	$9.3875e-012$	$2.44899e-018$	$6.60034e-030$
R.Time/Sec		5.152376	6.354012	8.521886	12.186823

Table (2): Comparison results for different value of N and N_* for boundary conditions in type II

t	Exact	Present Method for $N=5$			
		$N_*=4$	$N_*=7$	$N_*=10$	$N_*=15$
0.0	1.0	1.0	1.0	1.0	1.0
0.1	1.0001	1.0029756	1.0001068	1.000100004	1.0001
0.2	1.0016	1.0064499	1.0016115	1.001600006	1.0016
0.3	1.0081	1.0145381	1.0081152	1.008100009	1.0081
0.4	1.0256	1.0333013	1.0256182	1.025600011	1.0256
0.5	1.0625	1.0711534	1.0625205	1.062500012	1.0625
0.6	1.1296	1.1388921	1.1296220	1.129600013	1.1296
0.7	1.2401	1.2497083	1.2401227	1.240100013	1.2401
0.8	1.4096	1.4191893	1.4096227	1.409600014	1.4096
0.9	1.6561	1.6653203	1.6561217	1.656100013	1.6561
1.0	2.0	2.0084853	2.0000199	2.000000012	2.0
L.S.E		$6.04119e-003$	$3.38285e-008$	$1.22779e-014$	$2.94004e-027$
R.Time/Sec		9.308898	10.193173	12.368205	17.787931

Example (3):

Consider a high-order nonlinear IFDEs of F-H Type, on the interval [0,1]:

$$ {}_0^C D_t^{3\alpha} y(t) - \frac{1}{2}\, {}_0^C D_t^{2\alpha} y(t) - e^t\, {}_0^C D_t^{\alpha} y(t) + y(t) = f(t) + \int_0^1 \left\{ (s\ \sinh(t)) \left[{}_0^C D_s^{\beta} y(s) \right]^3 \right\} ds $$

where

$$ f(t) = \frac{-2}{\Gamma(3-3\alpha)} t^{2-3\alpha} + \frac{1}{\Gamma(3-2\alpha)} t^{2-2\alpha} + \frac{2e^t}{\Gamma(3-\alpha)} t^{2-\alpha} + 1 - t^2 + \left[\frac{2}{\Gamma(3-\beta)} \right]^3 \left(\frac{\sinh(t)}{8-3\beta} \right) $$

With boundary condition:

If $0 < \alpha < \frac{1}{3}$ then $y(0) + y(1) = 1$

If $\frac{1}{3} < \alpha < \frac{2}{3}$ then $y(0) + y(1) = 1, y''(1) = -2$

While the exact solution is $y(t) = 1 - t^2$.

- For $\alpha = 0.2$ and $\beta = 0.1$,we apply the algorithm[AGTM-NLFH]by the above procedure to obtain the approximate function $\tilde{y}(t)$for the solution of consider problem

$$ \tilde{y}(t) = -1.000336808557622t^2 + 0.000229936144535t^{0.4} - 0.000326546842829t^{0.2} - 0.016771772497147t - 0.001012291069540t^{0.6} + 0.005876741959711t^{0.8} + 0.027178015119165t^{1.2} - 0.023703350718884t^{1.4} + 0.009640703396928t^{1.6} - 0.000821245067574t^{1.8} + 1.000023309066628 $$

- For $\alpha = 0.4$ and $\beta = 0.2$, and apply the algorithm comparison result $\tilde{y}(t)$ with the exact solution

$$ \tilde{y}(t) = -1.0\, t^2 + 0.000194264776350\, t^{0.4} - 0.000153139023994\, t^{0.8} + 0.000056257657854\, t^{1.2} - $$

$0.000058485326795\, t^{1.6} +$
0.999980550958292

The result in Table (3) shows the least square errors and running time for example (3) with different values of α , β , N and N_*.

Table (3): Comparison results for different value for different values of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{N}$ and $\boldsymbol{N_*}$

Fractional order	$(\boldsymbol{\alpha},\boldsymbol{\beta}) = (0.4,0.2)$			$(\boldsymbol{\alpha},\boldsymbol{\beta}) = (0.2,0.1)$		
$(\boldsymbol{N}, \mathbf{N}_*)$	N = 5			N = 10		
	$N_* = 6$	$N_* = 8$	$N_* = 10$	$N_* = 6$	$N_* = 8$	$N_* = 10$
Error	$1.487e-007$	$1.8249e-011$	$7.7214e-016$	$2.1616e-002$	$3.38955e-004$	$3.95557e-007$
R.Time/Sec	5.88102	6.008356	6.129227	862.220	886.923	912.144

6. Conclusion:

It is usually difficult to solve analytically the high order nonlinear integro-Fractional differential equations of Fredholm-Hammerstein type with the mixed conditions. For this purpose, the presented numerical method can be proposed.

In this study, we present a new numerical method to obtain approximating solution is generalized Taylor collocation method. This method modifies non-linear integro-Fractional differential equations of Fredholm-Hammerstein type into matrix equations.

It is observed that the method is most advantageous when the approximate can be very easily calculated by using computer programs, this technique is fast than the other methods. Several example reveal that the methods are quite accurate and efficient.

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