On Sum Element Ideal Graphs

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A R T I C L E  I N F O

Article History:
Received: 9/2/2017
Accepted: 23/7/2017
Published: 28/12/2017

Keywords:
Zero divisor graph
Element ideal graph
Connectivity.

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A B S T R A C T

Let R be a commutative ring with identity and let x be an element of R. The element ideal graph \( \Gamma_x(R) \) is a graph whose vertex set is the set of non-trivial ideals of R and two distinct vertices I and J are adjacent if and only if \( x \in IJ \). In this paper, we consider a new kind of graph associated with R denoted by \( \gamma_x(R) \) whose vertex set is the set of non-trivial ideals of R and two distinct vertices I and J are adjacent if and only if \( I+J \) is a vertex of \( \Gamma_x(R) \). We explore some properties of this kind of graph.

1. INTRODUCTION

The rings considered in this paper are finite commutative with identity. The idea of associating a graph to a ring was initiated by (Beck, 1988) and consequently several researchers have done interesting and enormous work on zero divisor graphs of rings, such as: Akbari, Kami, Mohammadi and Moradi (Akbari, 2009), Akbari, Maimani and Yassemim (Akbari, 2003), Anderson and Badawi(Anderson, 2013), Anderson and Livingston(Anderson, 1999) and Anderson and Naseer(Anderson, 1993)

For an excellent and inspiring survey of the research work done in the area of zero divisor graphs in commutative rings, the reader is referred to (Beck, 1988).

Let R be a ring which is not an integral domain. Let \( Z(R) \) denote the set of all zero-divisors of R. Recall from (Beck, 1988) that the zero divisor graph \( \Gamma(R) \) of a ring R is a simple undirected graph with vertex set \( Z(R)^* \) and two distinct vertices x, y are adjacent in \( \Gamma(R) \) if and only if \( xy = 0 \).

Recall from (Behboodi, 2011) that an ideal I of R is said to be an annihilating ideal if \( rI = (0) \) for some \( r \in R \setminus \{0\} \). As in (Behboodi, 2011), we denote by \( A(R) \), the set of all annihilating ideals of R and by \( A(R)^* \), the set of all non-zero annihilating ideals of R. The annihilating ideal graph of a ring R, denoted by \( AG(R) \) is a simple undirected graph whose vertex set is \( A(R)^* \) and two distinct vertices I, J are adjacent in this graph if and only if \( IJ = (0) \). The concept of annihilating ideal graph of a ring was introduced by Behboodi and Rakeei (Behboodi, 2011). Many interesting and inspiring theorems proved in ((Behboodi and Rakeei, 2011), Aalipour, Akbari and Nikandish, (Aalipour, 2012) and Badawi (Badawi, 2014).
Recall from Abdul-Qadr (Abdul-Qadr and Shuker, 2015) that the element ideal graph of a ring $R$, denoted by $\Gamma_x(R)$, is a connected graph whose vertex set is the set of non-trivial ideals of $R$ and two vertices $I$ and $J$ are adjacent if and only if $x \in IJ$. The concept of element ideal graph of a ring was introduced by Abdul-Qadr in 2015 and many interesting theorems have been proved.

The aim of this paper is to study some properties of Sum element ideal graph and we investigate the interplay between the graph theoretic properties of $\gamma_x(R)$ and the ring theoretic properties of $R$.

From now on, we shall use the symbol $I-J$ to denote for two adjacent ideal vertices $I$ and $J$. Also we shall use the symbol $\text{Max}(R)$ to denote the set of all maximal ideals of $R$.

1. BACKGROUND

In this section, we state some definitions and theorems that we need in our work.

**Definition 1.1:** The element ideal graph $\Gamma_x(R)$ is a graph whose vertex set is the set of non-trivial ideals of $R$ and two vertices $I$ and $J$ are adjacent if and only if $x \in IJ$. (Abdul-Qadr and Shuker, 2015)

According to (CHARTLAND and LESNIAK, 1986), we will need the following definitions and results.

**Definition 1.2:**
1. The degree of a vertex $v$ in a graph $G$ is the number of edges of $G$ incident with $v$.
2. A graph $G$ is planar if it can be embedded in the plane.
3. A graph $G$ is complete if every two of its vertices are adjacent.
4. A vertex $u$ is said to be connected to a vertex $v$ in a graph $G$ if there exists a $u-v$ path in $G$. A graph $G$ is connected if every two of its vertices are connected.

5. The distance $d(u,v)$ between a pair of vertices $u$ and $v$ of the connected graph $G$ is the minimum of the lengths of the $u-v$ path.
6. The diameter of a connected graph $G$ is defined by $\text{diam}G = \max_{u,v \in V(G)} d(u,v)$.
7. The girth of a graph $G$ is the length of the shortest cycle contained in the graph $G$.

**Theorem 1.3:** (Kuratowsky Theorem)
A graph $G$ is planar if and only if it does not contain a graph homomorphic to $K_5$ or $K(3,3)$.

According to (Abdul-Qadr and Shuker, 2015) we will need the following definitions and results.

**Proposition 1.4:** If $I$ and $J$ are adjacent ideal vertices in $\Gamma_x(R)$, and $K$ is an ideal containing $J$, then $K$ is an ideal vertex of $\Gamma_x(R)$ and $I$ is adjacent to $K$ in $\Gamma_x(R)$.

**Lemma 1.5:** All vertices of $\Gamma_x(R)$ contains $x$.

**Theorem 1.6:** If $\Gamma_x(R)$ is non-empty, then its vertex set contains a maximal ideal of $R$. Moreover, if $I$ is an ideal vertex of $\Gamma_x(R)$ which is not maximal ideal, then $|\Gamma_x(R)| > 1$.

**Proposition 1.7:** The co-maximal ideal vertices of $\Gamma_x(R)$ are adjacent.

**Lemma 1.8:** Every proper ideal of a ring $R$ contained in a maximal ideal of $R$.

2. SUM ELEMENT IDEAL GRAPHS

We start this section by the following definition.

**Definition 2.1:** The Sum element ideal graph $\gamma_x(R)$ is undirected graph whose vertex set is the set of non-trivial ideals of $R$ and two distinct vertices $I$ and $J$ are adjacent if and only if $I+J$ is a vertex of $\Gamma_x(R)$.

Now, we consider some properties of the sum element ideal graph with providing some examples.

**Proposition 2.2:** If $I$ and $J$ are two distinct ideal vertices of $\Gamma_x(R)$ such that $I+J \neq R$, then they are adjacent in $\gamma_x(R)$. 
Proof: Since I is an ideal vertex of \( \Gamma_x(R) \), then there exists a non-trivial ideals K \( \neq I \) adjacent to I in \( \Gamma_x(R) \). If I+J \( \in \{I, J, K\} \), then I+J is an ideal vertex of \( \Gamma_x(R) \). If I+J \( \in \{I, J, K\} \), then \( x \in IK \subseteq (I+J)K \). This means that I+J is an ideal vertex of \( \Gamma_x(R) \) adjacent to K. Hence I and J are adjacent ideal vertices in \( \gamma_x(R) \).

Corollary 2.3: If P: I, K, J is an open path in \( \Gamma_x(R) \), then I and J are adjacent in \( \gamma_x(R) \).

Proof: Let P: I, K, J be an open path in \( \Gamma_x(R) \). Then \( x \in IK \) and \( x \in JK \). It follows that \( x \in (I+J)K \). Since P: I, K, J is an open path in \( \Gamma_x(R) \), I and J are not adjacent in \( \Gamma_x(R) \). Hence by Proposition 1.7, I+J \( \neq R \). From Proposition 2.2, I and J are adjacent in \( \gamma_x(R) \).

Example 1: Consider the ring \( Z_{18} \). The graphs \( \Gamma_{12}(Z_{18}) \) and \( \gamma_{12}(Z_{18}) \) are shown below:

![Fig.2.1: The graph \( \Gamma_{12}(Z_{18}) \)](image)

![Fig.2.2: The graph \( \gamma_{12}(Z_{18}) \)](image)

Clearly, (3) and (6) are ideal vertices of \( \Gamma_{12}(Z_{18}) \), (3)+(6) \( \neq Z_{18} \) and the ideal vertices (3) and (6) are adjacent in \( \gamma_{12}(Z_{18}) \).

The following result illustrates the relation between vertex sets of \( \Gamma_x(R) \) and \( \gamma_x(R) \).

Proposition 2.4:
1. The vertex set of \( \Gamma_x(R) \) is a subset of vertex set of \( \gamma_x(R) \).
2. The ideal I of R is an ideal vertex of \( \Gamma_x(R) \) if and only if every ideal J \( \neq (0) \) of R that contained properly in I, is adjacent to I in \( \gamma_x(R) \).

Proof: Since I is an ideal vertex of \( \Gamma_x(R) \), then there exists a non-trivial ideals K \( \neq I \) adjacent to I in \( \Gamma_x(R) \). If I+J \( \in \{I, J, K\} \), then I+J is an ideal vertex of \( \Gamma_x(R) \). If I+J \( \in \{I, J, K\} \), then \( x \in IK \subseteq (I+J)K \). This means that I+J is an ideal vertex of \( \Gamma_x(R) \) adjacent to K. Hence I and J are adjacent ideal vertices in \( \gamma_x(R) \).

Proposition 2.5:
1. The co-maximal ideals of \( R \) are not adjacent in \( \gamma_x(R) \).
2. Every maximal ideal vertex of \( \gamma_x(R) \) is an ideal vertex of \( \Gamma_x(R) \).

Proof: Let M and N be co-maximal ideals of \( R \). Since \( \Gamma_x(R) \) has no trivial ideal vertices, then \( M+N=R \) is not an ideal vertex of \( \Gamma_x(R) \). Thus M and N are not adjacent in \( \gamma_x(R) \).

Example 2: Let M be a maximal ideal vertex of \( \gamma_x(R) \). Then there exists an ideal vertex I \( \neq M \) adjacent to M in \( \gamma_x(R) \). This yields that I+M is an ideal vertex of \( \Gamma_x(R) \). Consequently, I+M \( \neq R \). Since M is a maximal ideal of R, then M=M+I. Thus M is an ideal vertex of \( \Gamma_x(R) \).

The following result explores some characteristics of the ideal (x) of R.

Theorem 2.6: Let \( x \neq 0,1 \).
1. The principal ideal (x)=R or (x) is a maximal ideal if and only if \( \gamma_x(R) = \emptyset \).
2. (x) is adjacent to all maximal ideal vertices of \( \gamma_x(R) \).
3. \( \deg(x) \geq |\text{Max}(R)| \).

Proof: Suppose that (x)=R or is a maximal ideal. If we assume that \( \gamma_x(R) \neq \emptyset \), then there exist two distinct non-trivial ideals I and J of R such that I+J is an ideal vertex of \( \Gamma_x(R) \). It follows that there exists a non-trivial ideal K different from I+J such that \( x \in (I+J)K \). This gives that (x)\( \subseteq (I+J)K \). Since (I+J)K \( \subseteq (I+J) \cap K \), then (x)\( \subseteq (I+J) \) and (x)\( \subseteq K \). If (x)=R, then R\( \subseteq K \). This contradicts that K is a non-trivial ideal. If (x) is a maximal ideal, then (x)=I+J=K. This
contradicts that I+J and K are distinct ideal vertices of \( \Gamma_X(R) \). Therefore \( \gamma_X(R) = \emptyset \).

**Conversely:** Suppose that \( \gamma_X(R) = \emptyset \). If we assume that \( (x) \neq R \) is not a maximal ideal of R, then by Proposition 2.5 there exists a maximal ideal M of R adjacent to \( (x) \) in \( \gamma_X(R) \). This contradicts the fact that \( \gamma_X(R) = \emptyset \). Thus either \( (x) = R \) or \( (x) \) is a maximal ideal.

2. Let \( M \in \gamma_X(R) \) be a maximal ideal of R. From Theorem 2.6, \( (x) \neq M \) and \( (x) + M \neq R \). From Proposition 2.5, \( (x) + M = M \) is an ideal vertex of \( \Gamma_X(R) \). Thus \( (x) \) is adjacent to \( M \) in \( \gamma_X(R) \).

3. The proof follows from the first part of the theorem.

**Example 2:** In the following graph, the ideal vertex \((12)\) is adjacent to both maximal ideals \((2)\) and \((3)\) of R, the co-maximal ideals \((2)\) and \((3)\) are not adjacent and every ideal vertex is adjacent to at least one of maximal ideals \((2)\) and \((3)\).

![Fig.2.3: The graph \( \gamma_{12}(Z_{36}) \)](image)

The next result shows the lower bound of the clique of \( \gamma_X(R) \).

**Theorem 2.7:** If \( \Gamma_X(R) \) contains a chain of ideal vertices of length \( n \geq 1 \), then \( \text{cl}(\gamma_X(R)) \geq n \). Moreover, if \( \gamma_X(R) \) is a planar graph, then \( n \leq 5 \).

**Proof:** Let \( I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \) be a chain of ideal vertices of \( \Gamma_X(R) \). Since \( I_i + I_j = I_{i+j} \in V(\Gamma_X(R)) \), for every \( i, j = 1, 2, \ldots, n \) with \( i \neq j \), then \( I_i \) and \( I_j \) are adjacent ideal vertices in \( \gamma_X(R) \), for every \( i, j = 1, 2, \ldots, n \). This means that \( \gamma_X(R) \) contains a complete subgraph \( G = K_n \) whose vertices are \( I_1, I_2, \ldots, I_n \). Thus \( \text{cl}(\gamma_X(R)) \geq n \). Suppose that \( \gamma_X(R) \) is a planar graph. Then by Kuratowski Theorem, the order of a graph \( G \) is less than 5.

**Example 3:** Consider the ring \( Z_{24} \). Then the clique number the following is 3:

![Fig.2.4: The graph \( \gamma_{18}(Z_{24}) \)](image)

The next result illustrates the completeness of \( \gamma_X(R) \) under a certain condition.

**Proposition 2.8:** If the vertex set of \( \gamma_X(R) \) is totally ordered, then \( \gamma_X(R) \) is a complete graph.

**Proof:** Let I and J be any two distinct ideal vertices of \( \gamma_X(R) \). Since the vertex set of \( \gamma_X(R) \) is totally ordered, either \( I \subseteq J \) or \( J \subseteq I \). It follows that either \( I + J = J \) or \( I + J = I \). From both cases we obtain that \( I + J \) is an ideal vertex of \( \Gamma_X(R) \). Thus I and J are adjacent in \( \gamma_X(R) \). Hence \( \gamma_X(R) \) is a complete graph.

The next result shows a relation between \( \Gamma_X(R) \) and \( \gamma_X(R) \).

**Theorem 2.9:**

1. If R is a local ring, then \( \Gamma_X(R) \) is a subgraph of \( \gamma_X(R) \).
2. If R is a local ring, then the graphs \( \Gamma_0(R) \) and \( \gamma_0(R) \) are complete identical graphs whose vertices are all non-trivial ideals of R.
3. If R is not a local ring, then \( \Gamma_X(R) \neq \gamma_X(R) \) for every \( x \) in which \( \Gamma_X(R) \neq \emptyset \).

**Proof:**

1. Let I—J be an edge in \( \Gamma_X(R) \). Then \( x \in I \cap J \). It follows that:

\[
 x \in (I+J)I \quad (1)
\]

Since R is a local ring, then \( I+J \neq R \). From (1) we obtain that I+J is an ideal vertex of \( \Gamma_X(R) \). This yields that I—J is an edge in \( \gamma_X(R) \). Thus \( \Gamma_X(R) \) is a subgraph of \( \gamma_X(R) \).
2. Let I and J be any two distinct non-trivial ideals of R. Clearly, I+J≠R because R is a local ring. Since 0∈IJ, then I and J are adjacent ideal vertices in $\Gamma_0(R)$. This yields that $\Gamma_0(R)$ is a complete graph and every non-trivial ideals of R is an ideal vertex of $\Gamma_0(R)$. For a special case, I+J is an ideal vertex of $\Gamma_0(R)$. Thus I and J are adjacent ideal vertices in $\gamma_0(R)$. Hence $\gamma_0(R)$ is a complete graph whose vertices are all non-trivial ideals of R. Obviously, The graphs $\Gamma_0(R)$ and $\gamma_0(R)$ are identical graphs.

3. Since R is not a local ring, then R has at least two ideals say M and N. From Proposition1.7 and Proposition2.5 we see that M and N are adjacent in $\gamma_x(R)$ but not in $\gamma_x(R)$.

The next result shows that R is a local ring by using a property of $\gamma_x(R)$.

**Proposition2.10:** If $\gamma_x(R)$ is a complete graph, then R is a local ring.

**Proof:** Suppose that $\gamma_x(R)$ is a complete graph. From Preposition2.5, the co-maximal ideals are not adjacent in $\gamma_x(R)$. Thus the completeness of $\gamma_x(R)$ leads to the existence only one maximal ideal of R. Hence R is a local ring.

The inverse of above theorem follows from the second part of Theorem2.9.

**Example4:** Consider the ring $\mathbb{Z}_{32}$. Then the graph $\Gamma_16(\mathbb{Z}_{32})$ is:

![Graph](image)

**Fig.2.5:** The graph $\gamma_16(\mathbb{Z}_{32})$

Obviously, R is a local ring and the graph $\Gamma_16(\mathbb{Z}_{32})$ is complete.

### 3. Connectivity of $\gamma_x(R)$

In this section we investigate the connectedness, the diameter and the girth of the sum element ideal graph.

The following result compute the distance between any two maximal ideals of R in $\gamma_x(R)$.

**Proposition3.1:** For every two distinct maximal ideals M and N of R in the vertex set of $\gamma_x(R)$, $d(M,N)=2$.

**Proof:** Let M, N∈Max(R) be distinct ideal vertices in $\gamma_x(R)$. It is Obvious from Proposition2.5 that M and N are adjacent in $\gamma_x(R)$ but not in $\gamma_x(R)$. Thus $P: I, (x), J$ is an I-J path in $\gamma_x(R)$. Hence $d(M,N)=2$.

The next result demonstrates the connectedness and the diameter of the element ideal graph.

**Theorem3.2:** Let $\gamma_x(R) \neq \emptyset$. Then $\gamma_x(R)$ is connected and $\text{diam}\gamma_x(R) \leq 4$.

**Proof:** Suppose that I and J be any two distinct ideal vertices of $\gamma_x(R)$. From Theorem2.6 we have $\gamma_x(R)=\emptyset$. We give the following cases for x, I and J:

**Case1:** Let $x=0$. Obviously, 0∈ST for every two non-trivial distinct ideals S and T of R. This gives that:

$V(\Gamma_0(R))=\{I: I \text{ is a non-trivial ideal of } R\}$  \(1\)

Here we have two subcases for I and J:

**Case1.1:** Let I, J∈Max(R). Then by Proposition2.5, I and J are not adjacent ideal vertices in $\gamma_0(R)$. Then there exists non-trivial ideals K and L of R adjacent to I and J respectively. Again from Proposition2.5, K and L are not maximal ideals of R, therefore $K+L\neq R$. From (1) we obtain that I+J is an ideal vertex of $\Gamma_0(R)$. Thus I and J are adjacent in $\gamma_0(R)$.
Case 1.2: Let \( \{I, J\} \notin \text{Max}(R) \). Then \( I+J \notin R \). It follows from (1) that \( I+J \) is an ideal vertex of \( \Gamma_0(R) \). Thus \( I \) and \( J \) are adjacent in \( \gamma_0(R) \).

Case 2: Let \( x \neq 0 \). We have four subcases for \( I \) and \( J \):

Case 2.1: Let \( I, J \in \text{Max}(R) \). It has been clarified through the steps of the proof of Proposition 3.1 that there is a path between \( M \) and \( N \) in \( \gamma_x(R) \) of length 2.

Case 2.2: Let \( I \in \text{Max}(R) \) and \( J \in \text{Max}(R) \). From Proposition 2.5 there exists \( M \in \text{Max}(R) \) such that \( I \) and \( J \) are adjacent to \( (x) \) and \( M \) respectively. If \( M=I \), then \( P: I, J \) is an \( I \)-\( J \) path in \( \gamma_x(R) \). If \( M \neq I \), then \( P: I, (x), M, J \) is an \( I \)-\( J \) path in \( \gamma_x(R) \).

Case 2.3: Let \( J \in \text{Max}(R) \) and \( I \in \text{Max}(R) \). By the same way of Case 2.2, an \( I \)-\( J \) path exists in \( \gamma_x(R) \).

Case 2.4: Let \( I, J \in \text{Max}(R) \). Then by Proposition 2.5 there exist \( M, N \in \text{Max}(R) \) such that \( I \) and \( J \) are adjacent to \( M \) and \( N \) in \( \gamma_x(R) \) respectively. If \( M=N \), then \( P: I, M, J \) is an \( I \)-\( J \) path in \( \gamma_x(R) \). Suppose that \( M \neq N \). By Theorem 2.6, \( M \) and \( N \) are adjacent to \( (x) \) in \( \gamma_x(R) \).

From each case we obtained a path between \( I \) and \( J \) in \( \gamma_x(R) \). Thus \( \gamma_x(R) \) is a connected graph.

From above cases we have shown that the distance between every two ideal vertices \( I \) and \( J \) is at most 4. Thus \( \text{diam}(\gamma_x(R)) \leq 4 \).

In the next result we find the girth of \( \gamma_x(R) \).

Theorem 3.3: If \( \gamma_x(R) \) contains an edge of non-maximal ideal terminals \( I \) and \( J \) with \( I \notin J \) and \( J \notin I \), then the girth of \( \gamma_x(R) \) is equal to 3.

Proof: Suppose that neither \( I \subseteq J \) nor \( J \subseteq I \). This implies that neither \( I+J=J \) nor \( I+J=I \). Obviously, \( I+I+J=J+I+J=I+J \in \Gamma_x(R) \). This means that \( I \) and \( J \) are adjacent to \( I+J \) in \( \gamma_x(R) \). Thus \( C:I, I+J, J, I \) is a cycle of length three in \( \gamma_x(R) \). Thus the girth of \( \gamma_x(R) \) is equal to three.

Example 5: Consider the sum element ideal graph \( \gamma_{12}(Z_{36}) \) as follows:

![Fig.3.1: The graph \( \gamma_{12}(Z_{36}) \)](image)

It's clear from the above figure that \( \gamma_{12}(Z_{36}) \) is a connected graph has diameter and girth 2 and 3 respectively.

REFERENCES


