Zero-Hopf Bifurcation in the Rössler’s Second System

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ABSTRACT

This paper is devoted to study the zero-Hopf bifurcation of the Rössler's second system. We characterize the parameters for which a zero-Hopf equilibrium point takes place at each point. We prove that there are three one-parameter families exhibiting such equilibria. The averaging theory of the first order is also applied to prove the existence of one periodic orbit bifurcating from the zero-Hopf equilibrium at the origin. Here, to visualize this, FireFlies software is used.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

We consider the 3-dimensional system which is given by Rössler (Roössler, 1979):

\[
\begin{align*}
x &= x - z - xy, \\
y &= -ay + x^2, \\
z &= b(cx - z),
\end{align*}
\]

(1)

where a, b and c are real parameters. This system is different from the classical system commonly referred to as Rössler system, henceforth here we call (1) the Rössler's Second System. System (1) has an equilibrium at the origin for all values of the parameters and a pair of symmetrically located equilibria \(A_{\pm}(\pm \sqrt{a(1-b)}, 1-c, \pm \sqrt{a(1-b)})\) when \(a(1-b) > 0\). When all the parameters are positive, the stability of equilibrium points and limit cycles as well as the occurrence of the Hopf bifurcation at the origin and the nontrivial equilibrium points are studied in (Salih, 2009) and (Amen, 2009), respectively. In (Kokubu, 2004), Kokubu and Roussarie have performed a numerical simulation to obtain chaotic dynamics when \((a, b, c) = (0.1, 0.08, 0.125)\)

A zero-Hopf equilibrium point is an equilibrium point of a three-dimensional autonomous differential system which has a zero eigenvalue and a pair of purely imaginary eigenvalues. When an infinitesimal periodic orbit bifurcates from the equilibrium point, such a kind of bifurcation is called zero-Hopf bifurcation. This type of bifurcation has been analysed by Guckenheimer (Guckenheimer, 1981), Guckenheimer and Holmes (Guckenheimer, 2014), Han (Han, 1998), Kuznetsov (Kuznetsov, 2004) and Scheurle and Marsden (Scheurle, 1984). It has been shown that, from the isolated zero-Hopf equilibrium
point some complicated invariant sets could be bifurcated under some conditions. In some cases, the chaotic behavior has been obtained as we see in the work of Baldomá and Seara (Baldomá, 2006, 2008), Broer and Vegter (Broer, 1984), Champneys and Kirk (Champney, 2004) and Scheurle and Marsden (Scheurle, 1984).

FireFlies is a new software which is proposed by Merrison (Merrison-Hort, 2015) and it has a powerful approach to visualize the dynamical behavior for nonlinear models. In (Merrison-Hort, 2015), Merrison-Hort applied it to three modes such as: a 2D model of neuronal activity, the classical Lorenz system and a 15D model of three interacting biologically realistic neurons.

In this paper, the averaging theory of the first order is applied to the Rössler’s second system to study the possible limit cycles bifurcating from the equilibrium points of type zero-Hopf. Furthermore, FireFlies is used to visualize the appearance of a limit cycle of the system which occurs after the perturbation of parameters.

The main result of this work is the following. At the beginning, we try to study the conditions of the zero-Hopf equilibrium point of the Rössler’s second system. The first and second proposition show that there are three one-parameter families of the system exhibiting a zero-Hopf equilibrium located at the equilibrium points.

**Proposition 1.** There is only a one-parameter family of the Rössler’s second system for which the origin of coordinates is a zero-Hopf equilibrium point. Namely:

$$a = 0, b = 1$$ and $$c \in (1, \infty)$$ \hspace{1cm} (2)

Since system (1) is symmetric with respect to the involution $$(x, y, z) \rightarrow (-x, y, -z)$$, therefore when there is a zero-Hopf equilibrium point at the point $$A_+$$, is also a zero-Hopf equilibrium point at the point $$A_-$$. Our analysis will be only at the point $$A_+$$.

**Proposition 2.** The equilibrium point $$A_+$$ is a zero-Hopf equilibrium point if the following conditions are held:

$$a = c, b = 0$$ and $$c \in \left(0, \frac{2}{3}\right)$$. \hspace{1cm} (3)

We also apply the averaging theory described in Theorem 3 to the Rössler’s second system (1).

**Theorem 1.** Let $$(a, b, c) = (\alpha \delta, 1 + \beta \delta, c_i + \gamma \delta)$$ be with $$c_i > 1$$ and $$\delta$$ a sufficiently small parameter. If

$$\alpha \beta < 0,$$

then the Rössler’s second system (1) has a zero-Hopf bifurcation at the equilibrium point located at the origin of coordinates, and a periodic orbit is born at this equilibrium point when $$\delta = 0$$.

**Theorem 2.** Consider the three-dimensional Rössler’s second system (1) and let

$$a = c_i + \alpha \delta, b = \beta \delta, c = c_i + \gamma \delta$$ \hspace{1cm} (4)

where $$0 < c_i < \frac{2}{3}, (\alpha, \beta, \gamma) \neq (0, 0, 0)$$ and $$\delta$$ be a sufficiently small positive parameter. Using the averaging theory of the first order, we cannot find periodic orbits bifurcating from the zero-Hopf equilibrium point located at $$A_+$$ or $$A_-$$.

Thus, any information about the possible periodic orbits bifurcating from the zero-Hopf equilibria at $$A_+$$ or $$A_-$$ is not provided by the averaging theory of first order.
2. PROOF OF THE MAIN RESULTS

This section is devoted to proof the main results that we mentioned in the previous section.

Proof of Proposition 1. The characteristic polynomial \( p(\lambda) \) of the linearization of system (1) at the equilibrium point located at the origin is given by

\[
p(\lambda) = -\lambda^3 + T\lambda^2 + K\lambda + D,
\]

Where

\[
T = 1 - (a + b), \\
K = a + b(1 - (a + c)), \\
D = ab(1 - c).
\]

In order to obtain one zero and two pure imaginary eigenvalues, we suppose \( p(\lambda) = -\lambda \) (\( \lambda^2 + \omega^2 \)); \( \omega > 0 \). By comparing, the following relations are obtained:

\[
a + b = 1, \\
ab(1 - c) = 0, \\
a + b(1 - (a + c)) = -\omega^2.
\]

The only case that satisfies the above equations is \( a = 0, b = 1 \) and \( c = 1 + \omega^2 \). Thus, we have proved Proposition 1.

Proof of Proposition 2. The characteristic polynomial \( p(\lambda) \) of the linearization of system (1) at the equilibrium point located at \( A \), is given by

\[
p(\lambda) = -\lambda^3 + T\lambda^2 + K\lambda + D,
\]

where

\[
T = c - (a + b), \\
K = a(3a - (b + 2)), \\
D = 2ab(c - 1).
\]

Because we must have a zero-eigenvalue, it is necessary that

\[
2ab(c - 1) \leftrightarrow b = 0,
\]

and also, we must have two pure imaginary eigenvalues \( \pm i\omega; \omega > 0 \), in this case \( p(\lambda) \) must be written in the form \( p(\lambda) = -\lambda(\lambda^2 + \omega^2) \); \( \omega > 0 \). By comparing, the following relations are obtained:

\[
c - a = 0, \\
3a - 2 = -\omega^2.
\]

These equations imply that \( a=c \) and \( c \in (0, \frac{2}{3}) \). Thus, we have proved Proposition 2.

Proof of Theorem 1. If \((a, b, c) = (\alpha, \beta, \gamma, \delta) \) with \( \delta > 0 \) a sufficiently small parameter, then the Rössler’s second system (1) becomes

\[
\begin{align*}
\dot{x} &= x - z - xy, \\
\dot{y} &= -\alpha x + x^2, \\
\dot{z} &= (1 + \beta\delta)((c + \gamma\delta)x - z).
\end{align*}
\]

Doing the rescaling of variables \((x, y, z) = (\delta X, \delta Y, \delta Z) \) , system (7) in the new variables \((X, Y, Z) \) writes

\[
\begin{align*}
\dot{X} &= X - Z - \delta XY, \\
\dot{Y} &= \delta(-\alpha Y + X^2), \\
\dot{Z} &= (\delta \beta + 1)((c + \gamma\delta)X - Z).
\end{align*}
\]

Now, we shall transform the linear part at the origin of the differential equation (8) when \( \delta = 0 \) into its real Jordan normal form, that is as
We verify that this change of variable

\[(X, Y, Z) = P(U, V, W), \quad (9)\]

In the new variables \((U, V, W)\), system (8) becomes

\[
\begin{align*}
\dot{U} &= -\omega V + \frac{1}{\omega(\omega^2 + \omega^2)}((-(\beta \omega^3 + \beta \omega^2 + \beta \omega + \gamma \omega + \gamma)U + (\beta \omega - \gamma)V \\
&\quad - (\omega + 1)(\omega^2 + \omega + 1) UW - (\omega^2 + \omega + 1)VW)e - (\beta \gamma (\omega + 1)U + V)e^2), \\
\dot{V} &= \omega U + \frac{1}{\omega(\omega^2 + \omega^2)}(((\omega + 1)(\beta \omega^3 + \beta \omega^2 + \beta \omega + \gamma \omega + \gamma)U - (\omega + 1)(\beta \omega - \gamma)V \\
&\quad + (\omega + 1)UW + VW)e + (\beta \gamma (\omega + 1)((\omega + 1)U + V)e^2), \\
\dot{W} &= \epsilon(-\alpha W + 2(\omega + 1)UV + (\omega + 1)^2U^2 + V^2).
\end{align*}
\]

Now, we pass the differential system (10) to cylindrical coordinates \((r, \theta, W)\) defined by \(U = r \cos(\theta); V = r \sin(\theta)\) and \(W = W\), and after we introduce the new independent variable, and we obtain

\[
\frac{dr}{d\theta} = \frac{-r \epsilon}{\omega^2(\omega^2 + \omega^2)}(-\cos(\theta) \sin(\theta) \beta \omega^4 + \cos(\theta)^2 \beta \omega^3 + \cos(\theta)^2 \omega^3 W \\
&\quad - 2 \cos(\theta) \sin(\theta) \beta \omega^3 + 2 \cos(\theta)^2 \omega^2 W - 2 \cos(\theta) \sin(\theta) \beta \omega^2 - \cos(\theta) \sin(\theta)^2 \omega^2 \\
&\quad + \cos(\theta) \sin(\theta) \omega^2 W + 2 \cos(\theta)^2 \gamma \omega + 2 \cos(\theta)^2 \omega W - 2 \cos(\theta) \sin(\theta) \beta \omega \\
&\quad - 2 \cos(\theta) \sin(\theta) \gamma \omega + 2 \cos(\theta)^2 \gamma + 2 \cos(\theta)^2 W + \beta \omega^2 + \beta \omega - \gamma \omega - \gamma - W \\
&\quad + \epsilon \beta \gamma (-\cos(\theta) \sin(\theta) \omega^2 + 2 \cos(\theta)^2 \omega - 2 \cos(\theta) \sin(\theta) \omega + 2 \cos(\theta)^2 - \omega - 1)) \\
&= \epsilon F_{11}(r, \theta, W) + O(\epsilon^2)
\]

\[
\frac{dW}{d\theta} = \frac{\epsilon}{\omega}(\omega^2 r^2 \cos(\theta)^2 + 2 \omega r^2 \cos(\theta)^2 + 2 \omega^2 r^2 \cos(\theta) \sin(\theta) + 2 \omega^2 \cos(\theta) \sin(\theta) \\
&\quad - \alpha W + r^2) = \epsilon F_{12}(r, \theta, W) + O(\epsilon^2).
\]

Our previous system has the form of the differential equation (20) with \(t = 0, x = (r, W)\)

\[
f_1(t_0, W_0) = \frac{1}{2 \pi} \int_0^{2\pi} F_1(0, r_0, W_0) \theta d\theta = -\frac{r_0}{2 \alpha}(\beta + W_0) \\
f_2(t_0, W_0) = \frac{1}{2 \pi} \int_0^{2\pi} F_2(0, r_0, W_0) \theta d\theta = \frac{1}{2 \alpha} (\omega^2 t_0^2 + 2 \omega r_0^2 - 2 \alpha W_0 + 2 r_0^2).
\]
The system \( f_1(r_0,W_0) = f_2(r_0,W_0) = 0 \) has three solutions, one of them satisfies \( r_0 > 0 \), namely
\[
(r_0, W_0) = \left( \frac{\sqrt{-2(\omega^2 + 2\omega + 2)\beta\alpha}}{\omega^2 + 2\omega + 2}, -\beta \right).
\]
The Jacobian (23) at this point takes the value
\[
\frac{-\alpha\beta}{\omega^2},
\]
which is nonzero. For \( \dot{\omega} > 0 \) sufficiently small, Theorem 3 guarantees the existence of a periodic solution \((r(0,\dot{\omega}), W(0,\dot{\omega})) \rightarrow (r_0, W_0)\) when \( \dot{\omega} \rightarrow 0 \) (as we see in the following figure). Therefore,
\[
(U(\theta,\dot{\omega}) = r(\theta,\dot{\omega})\cos(\theta), (V(\theta,\dot{\omega}) = r(\theta,\dot{\omega})\sin(\theta), W(\theta,\dot{\omega}))
\]
is the periodic solution of system (10), pull back to system (8), also it has the periodic solution \(X(\theta), Y(\theta), Z(\theta)\) which obtained from (13) under the change of variable (9).
Since \((x(\theta), y(\theta), z(\theta) = (\dot{\omega}X(\theta), \dot{\omega}Y(\theta), \dot{\omega}Z(\theta))\), then system (7) has a periodic solution which tends to the origin when \( \dot{\omega} \rightarrow 0 \). Thus, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when \( \dot{\omega} \rightarrow 0 \).

**Proof of Theorem 2.** If the parameters satisfy equation (4) and after transforming the equilibrium point \( A_+ \) to the origin, the Rössler's second system (1) is written as follows
\[
\dot{x} = (\gamma \dot{\alpha} c_1) x - \sqrt{(\alpha \dot{\alpha} + c_1)(1-\gamma \dot{\gamma} - c_1)} y - z - xy,
\]
\[
\dot{y} = 2\sqrt{(\alpha \dot{\alpha} + c_1)(1-\gamma \dot{\gamma} - c_1)} x - (\alpha \dot{\alpha} + c_1) y + x^2,
\]
\[
\dot{z} = \dot{\alpha} \dot{\beta} ((\lambda \dot{\alpha} + c_1) x - z).
\]
(14)

By rescaling the variables \((x,y,z) = (\dot{\omega}X, \dot{\omega}Y, \dot{\omega}Z)\) system (14) becomes
\[
\dot{X} = (\gamma \dot{\alpha} c_1) X - \sqrt{(\alpha \dot{\alpha} + c_1)(1-\gamma \dot{\gamma} - c_1)} Y - Z - \dot{\alpha} \dot{X} Y,
\]
\[
\dot{y} = 2\sqrt{(\alpha \dot{\alpha} + c_1)(1-\gamma \dot{\gamma} - c_1)} X - (\alpha \dot{\alpha} + c_1) Y + \dot{\alpha} X^2 x
\]
\[
\dot{z} = \dot{\alpha} \dot{\beta} ((\lambda \dot{\alpha} + c_1) X - Z).
\]
(15)

When \( \dot{\omega} = 0 \), the linearized system of (15) at the origin is not of the real Jordan form *i.e* as
\[
\begin{pmatrix}
0 & 0 & -\sqrt{c_1(2-3c_1)} \\
\sqrt{c_1(2-3c_1)} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

For doing that, we consider the linear change of coordinates
\[
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix} = P
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix},
\]
(16)

Where
\[
P =
\begin{pmatrix}
0 & -\sqrt{c_1(1-c_1)} & c_1(2-3c_1) \\
\sqrt{c_1(2-3c_1)} & -c_1 & -(6c_1 - 4)\sqrt{c_1(1-c_1)} \\
0 & 0 & -c_1(2-3c_1)^2
\end{pmatrix}.
\]

Then in the new variables \((y_1, y_2, y_3)\), system (15) becomes

---

Fig. 1: FireFlies is applied to show that one limit cycle can bifurcate from the origin to system (1). Here, we chose specific parameters \( a = \dot{\omega}, b = 1 - \dot{\omega} \) and \( c = 1.1 + \dot{\omega} \) where \( \dot{\omega} \) is a sufficiently small parameter.
Writing the differential system (17) to cylindrical coordinates \((r, \theta, y_3)\) defining by \(y_1 = r \cos(\theta)\), \(y_2 = r \sin(\theta)\) and \(y_3 = y_3\). Then, we introduce \(\theta\) as the new independent variable to obtain:

\[
\frac{dr}{d\theta} = \frac{e}{2 \sqrt{c_1(1-c_1)(3c_1-2)^2(c_1-1)c_1}} \left( -2 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \cos(\theta)^2 \beta c_1^2 + 6 \sqrt{c_1(1-c_1)} \sin(\theta) \cos(\theta) \beta c_1^3 - 6 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \gamma \cos(\theta)^2 c_1 + 2 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \cos(\theta)^2 \beta c_1 + 6 \sqrt{c_1(1-c_1)} \sin(\theta) \gamma \cos(\theta) c_1^2 - 10 \sqrt{c_1(1-c_1)} \sin(\theta) \cos(\theta) \beta c_1^3 + 4 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \gamma \cos(\theta)^2 - 3 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \gamma c_1^2 + 3 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \alpha c_1^2 + 2 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \beta c_1 - 4 \sqrt{c_1(1-c_1)} \sin(\theta) \gamma c_1^2 \right)
\]

\[
\frac{d\theta}{d\theta} = \frac{e}{2 \sqrt{c_1(1-c_1)(3c_1-2)^2(c_1-1)c_1}} \left( -2 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \cos(\theta)^2 \beta c_1^2 + 6 \sqrt{c_1(1-c_1)} \sin(\theta) \cos(\theta) \beta c_1^3 - 6 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \gamma \cos(\theta)^2 c_1 + 2 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \cos(\theta)^2 \beta c_1 + 6 \sqrt{c_1(1-c_1)} \sin(\theta) \gamma \cos(\theta) c_1^2 - 10 \sqrt{c_1(1-c_1)} \sin(\theta) \cos(\theta) \beta c_1^3 + 4 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \gamma \cos(\theta)^2 - 3 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \gamma c_1^2 + 3 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \alpha c_1^2 + 2 \sqrt{c_1(1-c_1)} \sqrt{c_1(2-3c_1)} \beta c_1 - 4 \sqrt{c_1(1-c_1)} \sin(\theta) \gamma c_1^2 \right)
\]

\[
\frac{dy_3}{d\theta} = \frac{e \gamma}{(3c_1-2)^2 c_1} \left( \sqrt{c_1(1-c_1)} c_1 y_2 + (-6c_1^3 + 10c_1^2 - 4c_1) y_3 \right) + O(\varepsilon^2).
\]
We note that the previous system is written as a differential system of the form (20) with
t = \theta,
(3x r, y) = (0, R, \infty),
(0, R, \infty) \times T_2,
(30 z r, y) = (0, 30z r, y) = (0, 30z r, y)
\int_{0}^{2\pi} f(r, y) F(r, y) d\theta = \epsilon F_{12}(r, \theta, y_3) + O(\epsilon^2),
\frac{dy_3}{d\theta} = \frac{\beta \epsilon}{c_1(3c_1 - 2)^3} (\sqrt{c_1(1 - c_1)} \sin(\theta)r - 6c_1^2 y_3 + 10c_1 y_3 - 4y_3) + O(\epsilon^2),
\frac{d^2}{d\theta^2} = \epsilon F_{12}(r, \theta, y_3) + O(\epsilon^2).

We note that the previous system is written as a differential system of the form (20) with t = \theta,
x = (r, y)_1 \in (0, \infty) \times R, \quad T = 2\pi, \quad z = (r_0, y_30),
and F_1(0, r, y_3) = (F_{11}(0, r, y_3), F_{12}(0, r, y_3)). It is a normal form for applying the averaging theory described in Theorem 3 and easy computation shows that
\int_{0}^{2\pi} f_1(t_0, y_{30}) d\theta = \frac{-2\alpha_{c_1}(2 - 3c_1)}{2c_1(1 - c_1)(9c_1^2 - 12c_1 + 4)} (-18c_1^4
+ 42c_1^3 + 32c_1^2 + 8c_1)y_{30} + \sqrt{c_1(1 - c_1)}
(-3c_1 - c_1 + 3c_1 + 2\alpha - 2\gamma)), \quad (19)
f_2(t_0, y_{30}) = \frac{1}{2\pi} \int_{0}^{2\pi} f_2(0, r_0, y_{30}) d\theta
= \frac{2(c_1 - 1)\beta}{\sqrt{c_1(2 - 3c_1)}} y_{30}.

In system (19), \((f_1(t_0, y_{30}), f_2(t_0, y_{30})) = (0, 0)\) does not have any nontrivial solutions, therefore the averaging theory described in Theorem 3 does not provide any information.
about the possible periodic orbits bifurcating from the zero-Hopf equilibrium point.

1. CONCLUSION

This paper presents the zero-Hopf bifurcation analysis at the three equilibrium points of the Rössler's second system, which is a family of ordinary differential equations defined in $\mathbb{R}^3$ and depending on three parameters, $a, b$ and $c$. The origin equilibrium point is a zero-Hopf point if $a = 0, b = c = 1$ and the other two equilibrium points, $A_\pm$, are of type zero-Hopf when $b = 0, a = c$ and $0 < c < \frac{2}{3}$. Furthermore, a periodic orbit of the system is obtained which is bifurcate from the origin via averaging method of first order.

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APPENDIX

The Averaging Method for Periodic Orbits

The averaging method is a classical and useful computational technique for analysing nonlinear oscillations. It has been used by many authors to study the bifurcating periodic orbits from a zero-Hopf equilibrium point. Castellanos et al. (Castellanos, 2013), García et al. (García, 2014), Llibre (Llibre, 2014a) and Llibre et al. (Llibre, 2015) have used the averaging theory of first order to study the possible periodic orbits bifurcating from the zero-Hopf equilibrium points of the tritrophic food chain model, a slow-fast system with two slow variables and one fast variable, the Rössler system and the Chen-Wang differential system respectively. In these studies, two periodic orbits were the maximum number.

The second order of averaging theory is applied to study the existence of periodic orbits to a quadratic polynomial differential system in $\mathbb{R}^3$, a class of three dimensional autonomous quadratic polynomial differential systems of Lorenz-type and a three-differential system by Llibre et al. (Llibre, 2009), Llibre and Pérez-Chavela (Llibre, 2014b), and Llibre and Xiao in (Llibre, 2014c) respectively. In their work, the maximum number of periodic orbits was three. Furthermore, the first and second orders of averaging theory were used to study the bifurcating periodic orbits of the FitzHugh-Nagumo system and the Chua system by Euzébio et al. (Euzébio, 2015) and Llibre and Euzébio (Euzébio, 2017) respectively.

Averaging methods are useful tools for investigating the number of periodic orbits for some differential systems. Many researchers have devoted their effort to study the existence of periodic orbits via this method which has a long history as we see in the work of Marsden and Mc-Cracken (Marsden, 1976), Chow and Hale (Chow, 1982), Sanders et al. (Sanders, 2007), Buică and Llibre (Buică, 2004), Buică et al. (Buică, 2007) and references therein. Here, we present the basic results that we need for proving the bifurcating periodic orbits from the zero-Hopf equilibrium points of system (1) which is the first order averaging method.

We consider

$$\dot{x} = \partial_1 F_1(t, x) + \partial_2 F_2(t, x, \dot{y}), \quad x(0) = x_0 \tag{20}$$

with $x \in D$; where $D$ is an open subset of $\mathbb{R}^n$, $t \rightarrow 0$. Moreover, we assume that both $F_1(t, x)$ and $F_2(t, x, \dot{y})$ are $T$-periodic in $t$. We also consider in $D$ the averaged differential equation

$$\dot{y} = \partial y, \quad y(0) = x_0 \tag{21}$$

where

$$f(y) = \frac{1}{T} \int_0^T F_1(t, y)dt \tag{22}$$

Under certain conditions which are shown in the below theorem, equilibrium solutions of the
averaged equation (21) correspond roughly to $T$–periodic solutions of equation (20).

**Theorem 3.** Consider the two initial value problems (20) and (21). Suppose:

i. $F_1$ its Jacobian $\frac{\partial F_1}{\partial x}$, its Hessian $\frac{\partial^2 F_1}{\partial x^2}$, $F_2$ and Jacobian $\frac{\partial F_2}{\partial x}$ are defined, continuous and bounded by a constant independent of $\delta \in [0,\infty) \times D$ and $\delta \in (0,\delta_f]$.

ii. $F_1$ and $F_2$ are $T$–periodic in $t$ ($T$ independent of $\delta$).

Then the following statements hold.

1. If $p$ is an equilibrium point of the averaged equation (21) and

$$\det \left( \frac{\partial f}{\partial y} \right)_{y=p} \neq 0,$$

(23)

then there exists a $T$–periodic solution $\phi(t,\delta_f)$ of equation (20) such that $\phi(0,\delta_f) \to p$ as $\delta \to 0$.

2. The stability or instability of the limit cycle $\phi(t,\delta_f)$ is given by the stability or instability of the equilibrium point $p$ of the averaged system (21). In fact, the singular point $p$ has the stability behaviour of the Poincaré map associated to the limit cycle $\phi(t,\delta_f)$.

**REFERENCES**


